2016

MATHEMATICS

[Honours]

PAPER – VII

Full Marks : 90

Time : 4 hours

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

GROUP – A

(*Mathematical Probability*)

[*Marks* : 36]

1. Answer any one question : 15 × 1

(*a*) (*i*) Define moment generating function of a random variable X. The probability density function of a random variable X is

$$f(x)=\frac{1}{2\theta}e^{-\frac{|x-\theta|}{\theta}}, -\infty < x < \infty.$$

Find the moment generating function of X. Hence find mean and variance of X. 5

(*ii*) A and B play a game which must be either won or lost. If the probability that A wins any game is p, find the probability that A wins 'm' games before B wins 'n' games ($m, n \geq 1$). 5

(*iii*) A point X is chosen at random in the interval $a \leq x \leq b$ in such a way that the probability that it lies in any sub-interval is proportional to the length of the sub-interval. Find the distribution of the random variable X. 5

(*b*) (*i*) Show that

$$F(x)=\begin{cases} 0, & -\infty < x < 0 \\ 1-e^{-x}, & 0 \leq x < \infty \end{cases}$$

is a possible distribution function, and find the density function. 2 + 3 (*ii*) If X and Y are independent continuous variates then show that the density function of $U = X + Y$ is given by

$$\phi(u) = \int_{-\infty}^{\infty} f_x(\alpha) f_y(u-\alpha)\, d\alpha$$

where f_x and f_y are the probability density functions of X and Y respectively. 5

(*iii*) A mathematician carries two match boxes, one in each pocket. Each box contains n sticks. Whenever he needs, he chooses a box at random and draws a match from it. Find the probability that the second box will contain r matches when (A) the first box is just emptied (i.e. the last match is drawn from the first box), (B) the first box is found to be empty for the first time. 5

2. Answer any *two* questions : 8 × 2

(*a*) (*i*) A box contains 10 balls numbered 1 to 10. Balls are drawn from it successively without replacement. What is the probability that ball no. 5 is drawn at the fifth drawing ? 5

(*ii*) If χ_1^2, χ_2^2, are independent χ^2-variates having m and n degree of freedom respectively, find the distribution of $\frac{\chi_1^2}{\chi_2^2}$. 3

(*b*) (*i*) Prove that a Gamma distribution with parameters (α, p) $(\alpha > 0, p > 0)$ is positively skewed and leptokurtic. 4

(*ii*) Show, by Tchebycheff's inequality, that in 2,000 throws with a coin the probability that the number of heads lies between 900 and 1,100 is at least 19/20. 4

(*c*) (*i*) Define convergence in probability. If sequence of random variables $\{X_n\}$ and $\{Y_n\}$ converges in probability to X and Y respectively then prove that $\{X_n Y_n\}$ converges in probability to XY. 2 + 3

(*ii*) If X_1 and X_2 are two Binomial variates with parameters (n, p_1) and (n, p_2), find the distribution of their sum. 3

(*d*) (*i*) Prove that $E(X^2) \geq \{E(X)\}^2$. Hence deduce that the first absolute moment about the mean is at most equal to the standard deviation. 2 + 3

(*ii*) If the random variables X and Y are connected by the linear relation $aX + bY + c = 0$, then find the correlation coefficient between X and Y. 3

3. Answer any *one* question : 3 × 1

(*a*) Let X_1, X_2, X_3 be three independent $N(\mu, 1)$ variables. Prove that

$$P(X_1 - 2X_2 + X_3 > 0) = \frac{1}{2}.$$ 3

(*b*) Obtain the recurrence relation

$$\mu_{k+1} = \mu\left[k\mu_{k-1} + \frac{d\mu_k}{d\mu}\right]$$

for the Poisson distribution with parameter μ, where μ_k represents the kth order central moment. 3

4. Answer any *one* question : 2×1

(*a*) Two random variables X and Y have zero means and standard deviation 1 and 2 respectively. Find the variance of $X + Y$ if X and Y are uncorrelated. 2

(*b*) State 'Central limit' theorem. 2

GROUP – B

(*Statistics*)

[*Marks* : 27]

5. Answer any *one* question : 15×1

(*a*) (*i*) What do you mean by sampling distribution of a statistic ? Define consistent and unbiased estimator of a statistic. $2 + 2 + 2$

(*ii*) Show that sample mean follows normal distribution. 5

(*iii*) If S^2 denotes the sample variance then find $E(S^2)$. 4

(*b*) (*i*) What do you mean by confidence interval in connection to interval estimation of a statistic ? 3

(*ii*) State Neyman-Pearson lemma in connection to Best critical region. 4

(*iii*) Find the maximum likelihood estimate of σ^2 for a normal (m, σ) population, when m is known and m is unknown. 8

6. Answer any *one* question : 8 × 1

(*a*) (*i*) A certain pesticide is packed into bags by a machine. A random sample of 10 bags is drawn and the contents in there are found to have the following weights (in kg).

50, 49, 52, 44, 45, 48, 46, 45, 49, 45

Test if the average quantity packed can be taken to be 50 kg.

[Given that $P(t > 2{\cdot}262) = 0{\cdot}025$]. 4

(*ii*) Find the Kurtosis for following data and comment on the nature of the distribution. 3 + 1

Groups :	0-10	10-20	20-30	30-40
Frequency :	1	3	4	2

(*b*) What do you mean by critical region in connection with testing of hypothesis ? The measurements (in mm) of rainfall of a rainy season in a particular region in India are 9·42, 8·89, 10·63, 12·25, 11·86, 11·47, 9·97, 10·85, 12·19, 11·72. Is it resonable to believe (under 5 % level of significance) that the average rainfall in that region is 11·32 mm ? Assume that population of rainfall has normal distribution. Given that for 9 degrees of freedom $P(t > 2{\cdot}262) = 0{\cdot}025$. 2 + 6

7. Answer any *one* question : 4 × 1

(*a*) Prove that the distribution of the sample approximates to the distribution of the population if the size of the sample is large. 4

(*b*) A die was thrown 400 times and number six was obtained 80 times. Does the data justify the hypothesis that the die is unbiased ? [Given that $P(u > 1{\cdot}96)=0{\cdot}025$]. 4

GROUP – C

[Optional Paper–I]

(*Discrete Mathematics*)

[*Marks* : 27]

8. Answer any *one* question : 3 × 1

(*a*) Using generating functions solve the recurrence relation :
$a_n = 4a_{n-1} + 3$ for $n \geq 1$ and $a_0 = 2$. 3

(*b*) Show by mathematical induction that any postage of Rs. $n \geq$ Rs. 5 can be done by using just Rs. 2 and Rs. 5 stamps. 3

9. Answer any *two* questions : 12 × 2

(*a*) (*i*) Determine number of integers between 1 and 250 that are divisible by any of the integers 2, 3, 5, 7. 4

(*ii*) If $(A, \leq)$ and $(B, \leq)$ be two partially order sets then prove that $(A \times B, \leq)$ is partially order set with partial order $\leq$ defined by $(a, b) \leq (a', b')$ if $a < a'$ in A and $b \leq b'$ in B. 4

(*iii*) Show that a complete graph with n vertices consists of $\frac{n(n-1)}{2}$ edges. 4

(*b*) (*i*) Define Tree. Show that a connected graph with n vertices and $n-1$ edge is a Tree. 1 + 3

(*ii*) Let $\{A, B, C, S\}$ be the set of nonterminals with S being the starting symbol. Let $\{a, b, c\}$ be the set of terminals. Describe the language specified by the following set of production in set theoretic notation :
$\{S \to as, S \to bA, A \to aA, A \to a\}$ 4

(*iii*) Define finite state machines. Construct a finite state machine that adds two integers in binary representation. 1 + 3 (*c*) (*i*) State principle of inclusion-exclusion defined on finite set. Find the number of positive integer $\leq$ 300 and divisible by 2 or 3. 4

(*ii*) Prove that every chain $(L, \leq)$ is a distributive lattice. 3

(*iii*) Prove that a connected graph G with n vertices and e edges has unique circuit iff $n = e$. 5

[Optional Paper—II]

(*Mathematical Modelling*)

[*Marks* : 27]

8. Answer any *one* question : 15 × 1

(*a*) The Lotka Volterra competition model of two different species satisfies the differential equation

$$\frac{dx}{dt} = r_1 x - \alpha_1 xy, \frac{dy}{dt} = r_2 y - \alpha_2 xy$$

where r_1, r_2, α_1, and α_2 are positive constants with $x(0) = x_0$, $y(0) = y_0$ and $x(t)$, $y(t)$ are the population of different species at time t.

(*i*) Find its solution and interpret the system geometrically.

(*ii*) Find its equilibrium points and discuss their stability. (2 + 5) + (2 + 6)

(*b*) (*i*) Write down the basic assumptions of epidemic model with succeptibles (S) and infectives (I) (SI model). 4

(*ii*) Write down the differential equation of this model. 3

(*iii*) Find its solution and interpret the result. 8

9. Answer any *one* question : 8 × 1

(*a*) A biologist begins a laboratory study of a certain animal population at time $t = 0$, when the population consists of 18 individuals. On the basis of data collected during the experiments, the biologist finds that the differential equation,

$$\frac{dp}{dt} = \frac{1}{1000}(30 - P)P$$

describe the growth and regulation of the population. Solve this equation for the population size P as a function of t, and sketch the graph of the function. Is there an instant of time when the growth rate of the population is maximum? 8

(*b*) A Colony of birds has a stable population. Prior to this situation, the population has increased from an initial low level. When the population was 10,000, the proportionate birth rate was 50 % per year and the proportionate death rate was 10% per year. When the population was 20,000, the proportionate birth rate was 30% and the proportionate death rate was 20 %. A model of the population is based on the following assumptions :

(*i*) there is no migration and no exploitation.

(*ii*) the proportionate birth rate is a decreasing linear function of the population.

(*iii*) the proportionate death rate is an increasing linear function of the population.

Show that a model based on these assumptions and above data predicts that the population grows according to the logistic model and find the stable population size. Shooting of birds is now allowed at a rate of 20% of the population per year. Find the new equilibrium population. 8

10. Answer any *one* question : 4 × 1

(*a*) Discuss age-structured population model. 4

(*b*) For the model $\frac{dx}{dt} = \alpha x(1 - \frac{x}{k}) - Ex$ with $x(0) = k$ where α, E and k are constants. Determine $x(t)$ explicitly. Show that $x > k\left(1 - \frac{E}{\alpha}\right)$. If $E \leq \alpha$, then $x \to k\left(1 - \frac{E}{\alpha}\right)$ as $t \to \infty$ where as if $E > \alpha$, then $x \to 0$ as $t \to \infty$. 4

[Optional Paper–III]

(*Application of Mathematics in Finance and Insurance*)

[*Marks* : 27]

8. Answer any *one* question : 15 × 1

(*a*) (*i*) Calculate the amount available when Rs. 20,000 is invested for five years and the interest on it is compounded at 8% p.a. half yearly.

(*ii*) A person deposits Rs. 3000 at the end of each year for 5 years at 4% rate of interest. How much would he receive at the end of the 5th year ?

(*iii*) State the function of financial management. 4 + 4 + 7

(*b*) What do you mean by risk associated with any corporate security ? Strike out the differences between systematic and unsystematic risk factors. State the role of 'Beta' as a measure of market risk in any corporate security. 4 + 5 + 6

9. Answer any *one* question : 8 × 1

(*a*) Write a note on bond valuation.

(*b*) Branches of insurence business with particular reference to India.

10. Answer any *one* question : 4 × 1

(*a*) What are the different components of returns from a corporate security ?

(*b*) "Options and futures are zero-sum games"— What do you think by this statement.
