DISTANCE LEARNING MATERIAL

VIDYASAGAR UNIVERSITY

DIRECTORATE OF DISTANCE EDUCATION

MIDNAPORE- 721 102

M. SC. IN APPLIED MATHEMATICS WITH OCEANOLOGY & COMPUTOR PROGRAMMING

PART - I

Paper : V

Module No. : 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59 & 60

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-V : Group – A : Marks – 50

Module No. - 49
(Mechanics of Continuous Media)

STRUCTURE :

1.1 INTRODUCTION

In this module we have discussed about the basic concept of the continuum mechanics which is essential to study about Strain.

1.2 OBJECTIVE

The concept of continuum mechanics is essential to learn the properties of matter. In this module the students will learn the methods and techniques to discuss the Strain and how it acts on the continuous media.

1.3 Key Words : Continuum, Configuration, Strain, Deformation, Displacement Gradient, Strain Tensor, Small Deformation, Normal Strain, Shearing Strain, Cubical Dilatation, Uniform Dilatation.

1.1 Concept of Continuum :

It is a common knowledge that every physical object is made of molecules, atoms and even smaller particles. These particles are not continuously distributed over the object; microscopic observations reveal the presence of gaps (empty spaces) between particles. While studing the external effects on physical objects, these gaps may or may not be taken into consideration depending on the hypothesis made. The study that takes account of the existence of gaps is called *microscopic study*. On the other hand, the study that ignores the gaps and treats a physical object as a continuous distribution of matter is called *macroscopic study*. The subjects of solid and fluid mechanics are concerned mainly with macroscopic study.

We see various types of materials like milk, water, honey, blood, wood, stone, steel, glass and flesh. Every such material body occupying a region of space is composed of a number of discrete molecules separated from one another by empty space comparable with molecular size. The matter in the body is evidently discontinuous and each molecule in the body obeys the law of Newtonian mechanics.

Thus the macroscopic viewpoint adopted in the study of solid and fluid mechanics leads to the notion of a continuous medium, or briefly a continnum. By a continuum, we mean a hypothetical physical object in which the matter is continuously distributed over the entire objects. So, a continuum is defined to be a *continuously distributed matter completely filling up the region of space, it occupies, with no empty space so as to ensure that it possesses unique physical properties such as unique density, unique displacement, unique velocity at every point of the space*. These physical properties can now be expressed as continous functions of position and time.

1.5 Continuum Particle :

We can define the material point or the particle as the smallest piece of matter containing a large number of molecules within an infinitesimally small volume whose physical dimensions are so small that it may be regarded as concentrated at a point. Accordingly in continuum we can associate a material point with each and every spatial point of the region of space occupied by a continuum body.

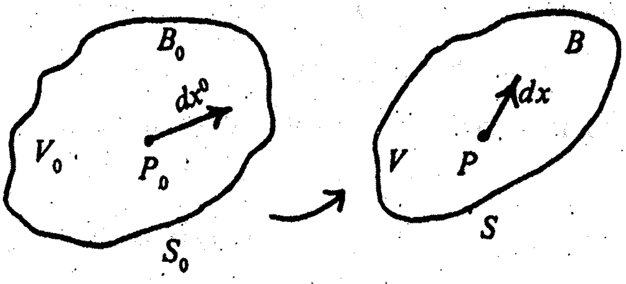
1.6 Configuration :

The complete specification of the set of positions of all material points of the body at a given time is called the configuration of the body at that time. This specification is the region of space of certain volume having a boundary surface. This region contains the continuous body at a given time.

1.7 Deformation :

Let us consider a continuum body of material points occupying at $t = 0$, a certain region of space B_0 consisting of volume V_0 and its surface S_0. When external forces are applied to the body, the material points of B_0 move so that at subsequent time t, they occupy some other region of space B consisting of volume V and surface S. Consequently there are changes in the positions of all the material points of the body. The body is then said to be deformed and the transformation of the body from its initial configuration to subsequent configuration is called *deformation.*

As a result of general deformation, a continuum body will change its configuration and orientation as well as its shape. Our first task is to separate that part of the general deformation which causes a change in configuration and orientation of the body from that part which causes a change in the shape of the body.



When the deformation is such that there are no changes in the relative positions of constituent material points of the continuum firmly bound together so that the length of any line joining any two material points does not change, then the deformation is a combination of translation and rotation about an axis at any point causing a change in configuration and orientation of the body only and is called rigid-body deformation and the body is called the rigid body.

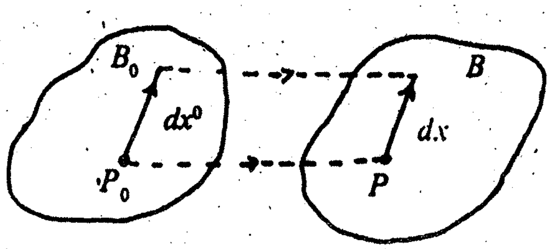
1.8 Strain Deformation :

When the deformation is such that there are changes in the relative positions of constituent material points of the continuum body so that the length and orientation of any line joining any two points changes then the deformation causes a change in the shape of the body only and is called strain deformation and the body is called deformable body.

The idea of strain is of relative rather than absolute change in the positions of material points: The existence of strain deformation depends on the occurrence of relative displacement of points in the medium w.r.t. each other.

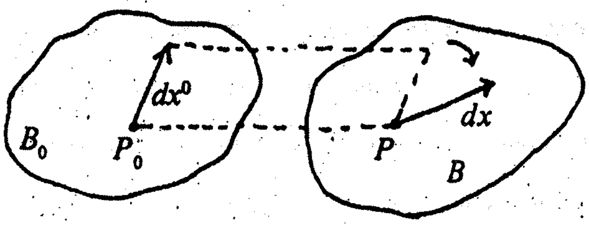
1.9 Translation :

When both the length and the orientation of the material arc dC are now preserved under the deformation, then such a deformation is called a rigid-translation or just a translation.



1.10 Rotation :

The effect of the deformation is just to change the orientation of the material arc dC without changing its length. Such a deformation is called a rigid rotation or just rotation. It is to be noted that, under a rigid rotation, the material arc dC may experience a translation as well.



1.11 Motion :

When a continuum body undergoes a deformation which continuously varies with time so that there is a continuing change in the configuration of the continuum with the passage of time, we say that continuum body is in motion.

1.12 Material Method or Lagrangian Method of Describing Deformation :

In this method we identify individual material points and describe the motion of each individual material point of fixed identity for all time by following its motion through out its course. Individual material points are change in two ways: as we pass from one material point to another and they change as time changes for a fixed material point, i.e., the physical properties of the individual material are considered as functions of time and of those data which identify the material points.

Each material point of the continuum is identified by the rectangular cartesian co-ordinates (X_1, X_2, X_3) of its position in its initial undeformed state. In this description the physical properties are assigned to these material points labelled by these co-ordinates at the initial configuration. All physical properties associated with this material point will then be functions of X_1, X_2, X_3 and time t. The primary quantity in this method is the position of the material point in the deformed state of the body at subsequent time t. If (x_1, x_2, x_3) be the rectangular cartesian co-ordinates of this position, then

$$x_i = x_i(X_1, X_2, X_3, t),\ i = 1,2,3 \ldots\ldots (1)$$

Above equations describes the motion of the material point completely in material method giving the subsequent position at time t. The co-ordinates X_1, X_2, X_3 are independent co-ordinates called material co-ordinates or Lagrangian co-ordinates, whereas x_1, x_2, x_3 are dependent co-ordinates called spatial co-ordinates.

1.13 Spatial Method or Eulerian Method :

In this method we identify the spatial points and describe the motion of the medium at each spatial point at different times without considering the whereabouts of individual material points. We focus our attention on a fixed spatial point in space occupied by different material points at different times and observe what changes of various properties are taking place at the spatial point. The spatial points, in a manner of speaking are endowed with physical properties and a material point is said to acquire these properties associated with the fixed spatial point when it happens to pass through that spatial point. In this method, physical properties changes in two ways: when we pass from one spatial point to another point and with time at a fixed spatial point.

If a material point which was at the position (X_1, X_2, X_3) in undeformed state at t=0 happens to occupy the position x_1, x_2, x_3 at subsequent time t, then co-ordinates x_1, x_2, x_3 identify the spatial point in the deformed state. Thus the physical properties will be functions of (x_1, x_2, x_3) and time t. In particular,

$$X_i = X_i(x_1, x_2, x_3, t),\ i = 1,2,3 \ldots\ldots (2)$$

The primary quantity in this method is the velocity of the material point. If v_1, v_2, v_3 be the components of the velocity of the material point occupying the position (x_1, x_2, x_3) in the deformed state at time t, then

$$v_i = v_i(x_1, x_2, x_3, t),\ i = 1,2,3 \ldots\ldots (3)$$

The velocity is measured at the current position (x_1, x_2, x_3). It is to be noted that the initial co-ordinates X_1, X_2, X_3 used in material method is irrelevant in spatial method. The symbols x_1, x_2, x_3 serve as the name of the place in space where material point resides. They are called spatial co-ordinates or Eulerian co-ordinates and they are independent.

Equations (1) and (2) are inverse to each other. For fixed time t, we have

$$x_i = x_i(X_1, X_2, X_3),\ i = 1,2,3 \ldots\ldots (4)$$

$$\text{and} \quad X_i = X_i(x_1, x_2, x_3),\ i = 1,2,3 \ldots\ldots (5)$$

These equations will only relate initial configuration to subsequent configuration and will naturally characterise the deformation.

1.14 Displacement :

The displacement u_i of a typical material point from its position (X_1, X_2, X_3) in the undeformed state at t=0 to its position (x_1, x_2, x_3) in deformed state at time t is defined by

$$u_i = x_i - X_i,\ i = 1,2,3 \ \ldots\ldots\ldots (6)$$

It should be noted that in the material method i.e., in the Lagrangian method u_i and x_i are regarded as functions of X_1, X_2, X_3 and t so that

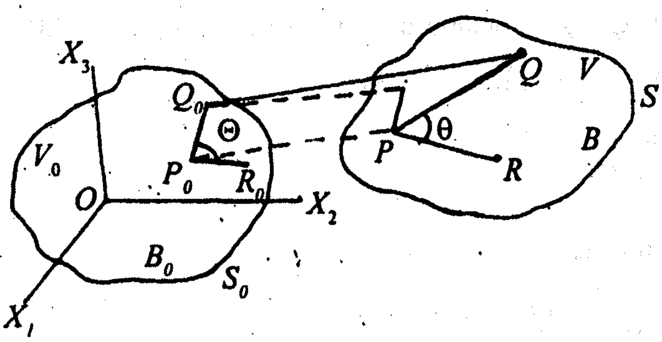
$$u_i(X_1, X_2, X_3, t) = x_i(X_1, X_2, X_3, t) - X_i \ \ldots\ldots\ldots (7)$$

In the spatial description i.e., in the Eulerian method, u_i and X_i are regarded as functions of x_1, x_2, x_3 and t so that

$$u_i(x_1, x_2, x_3, t) = x_i - X_i(x_1, x_2, x_3, t) \ \ldots\ldots\ldots (8)$$

1.15 Measures of Finite Strain Deformation : Finite Strain Tensor in Lagrangian Method :

We consider a change in the length of material line element i.e., the finite strain deformation from initial undeformed configuration (B_0, S_0, V_0) to the deformed configuration (B, S, V).



Let us consider a material element (line) P_0Q_0, joining a pair of neighbouring points P_0, Q_0 of length dL oriented in the direction (N_1, N_2, N_3) in the initial undeformed region B_0 at time t=0. If P_0 has co-ordinates (X_1, X_2, X_3) and Q_0 has co-ordinates $(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3)$ w.r.t. an orthogonal set of co-ordinate axes fixed in space, then

$$\begin{aligned} dL^2 &= (X_1 + dX_1 - X_1)^2 + (X_2 + dX_2 - X_2)^2 + (X_3 + dX_3 - X_3)^2 \\ &= dX_1^2 + dX_2^2 + dX_3^2 \\ &= \delta_{ij} dX_i dX_j \ \ldots\ldots\ldots\ldots\ldots\ldots (1) \end{aligned}$$

and $N_i = \frac{dX_i}{dL}, i = 1,2,3$

where δ_{ij} is the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

When the body undergoes deformation, the positions of these two points will be P and Q in the region B at time t. Let dl be the length of the new line lement PQ oriented in the direction (n_1, n_2, n_3) and the co-ordinates of P and Q are (x_1, x_2, x_3) and $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ respectively w.r.t. the same fixed set of co-ordinate axes. Then

$$dl^2 = (x_1 + dx_1 - x_1)^2 + (x_2 + dx_2 - x_2)^2 + (x_3 + dx_3 - x_3)^2$$
$$= dx_1^2 + dx_2^2 + dx_3^2$$
$$= \delta_{ij} dx_i dx_j \text{ .... (3)}$$

and $n_i = \frac{dx_i}{dl}, i = 1,2,3$ .... (4)

In material method or Lagrangian method, the deformation is characterised by equation

$$x_k = x_k (X_1, X_2, X_3, t) \text{ .... (5)}$$

$$\therefore dx_k = \frac{\partial x_k}{\partial X_j} dX_j + \frac{\partial x_k}{\partial t}$$

Since $x_k + dx_k$ are co-ordinates of Q at the same time t, so,

$$dx_k = \frac{\partial x_k}{\partial X_j} dX_j = \frac{\partial x_k}{\partial X_i} dX_i \text{ .... (6)}$$

Using (6) in (3), then we get

$$dl^2 = \delta_{ij} \frac{\partial x_i}{\partial X_p} dX_p \cdot \frac{\partial x_j}{\partial X_k} dX_k = \frac{\partial x_i}{\partial X_k} \cdot \frac{\partial x_i}{\partial X_p} dX_k dX_p \text{ .... (7)}$$

Hence, the difference $dl^2 - dL^2$ is the measure of change in length of a line element.

So, $dl^2 - dL^2 = \frac{\partial x_i}{\partial X_k} \cdot \frac{\partial x_i}{\partial X_p} dX_k dX_p - \delta_{ij} dX_i dX_j$

$$=\frac{\partial x_i}{\partial X_k}\cdot\frac{\partial x_i}{\partial X_p}dX_k dX_p-\delta_{kp}dX_k dX_p$$

$$=\left[\frac{\partial x_i}{\partial X_k}\cdot\frac{\partial x_i}{\partial X_p}-\delta_{kp}\right]dX_k dX_p$$

$$=2r_{kp}dX_k dX_p,\text{ say, }............(8)$$

where $$r_{kp}=\frac{1}{2}\left[\frac{\partial x_i}{\partial X_k}\cdot\frac{\partial x_i}{\partial X_p}-\delta_{kp}\right]$$

Also, $$\frac{dl^2-dL^2}{dL^2}=2r_{kp}\frac{dX_k}{dL}\cdot\frac{dX_p}{dL}=2r_{kp}N_k N_p............(10)$$

If u_i be the displacement of a material point from its position P_0 to P then

$$u_i=x_i-X_i..................(11)$$

If u_i+du_i be the displacement of the material point from its position Q_0 to Q then

$$u_i+du_i=(x_i+dx_i)-(X_i+dX_i)............(12)$$

or, $x_i-X_i+du_i=x_i-X_i+dx_i-dX_i$ (using (11))

or, $dx_i=du_i+dX_i$

$$\therefore\frac{\partial x_i}{\partial X_k}=\frac{\partial u_i}{\partial X_k}+\delta_{ik}...............(13)$$

and $$\frac{\partial x_i}{\partial X_p}=\frac{\partial u_i}{\partial X_p}+\delta_{ip}...............(14)$$

Hence from (9), it is clear that

$$r_{kp}=\frac{1}{2}\left[\left(\frac{\partial u_i}{\partial X_k}+\delta_{ik}\right)\left(\frac{\partial u_i}{\partial X_p}+\delta_{ip}\right)-\delta_{kp}\right]$$

$$=\frac{1}{2}\left[\frac{\partial u_p}{\partial X_k}+\frac{\partial u_k}{\partial X_p}+\frac{\partial u_i}{\partial X_k}\cdot\frac{\partial u_i}{\partial X_p}\right].........(15)$$

1.16 Change in the Angle Between Two Line Elements :

Next we consider the change in angle between two material line elements P_0Q_0 and P_0R_0 at P_0 inclined at an angle Θ, where $P_0Q_0(=dL)$ oriented in the direction (N_1,N_2,N_3) and $P_0R_0(=\delta L)$ oriented in the direction

(M_1, M_2, M_3) in the region B_0 where Q_0 and R_0 has the co-ordinates $(X_i + dX_i)$ and $(X_i + \delta X_i)$ respectively, then

$$\cos\Theta = \frac{dX_i}{dL}\cdot\frac{\delta X_i}{\delta L} \quad \text{............ (16)}$$

$$M_i = \frac{\delta X_i}{\delta L}, N_i = \frac{dX_i}{dL} \quad \text{................ (17)}$$

In the deformed state these line lements are PQ and PR respectively and θ is the angle between them. Let $PQ = dl$ and $PR = \delta l$ are oriented in the direction (n_1, n_2, n_3) and (m_1, m_2, m_3). If Q and R has co-ordinates $x_i + dx_i$ and $x_i + \delta x_i$, then

$$\cos\theta = \frac{dx_i}{dl}\cdot\frac{\delta x_i}{\delta l} \quad \text{............ (18)}$$

and $$m_i = \frac{\delta x_i}{\delta l},\ n_i = \frac{dx_i}{dl}$$

Now from (6), it is clear that

$$\delta x_k = \frac{\partial x_k}{\partial X_j}\delta X_j = \frac{\partial x_k}{\partial X_i}\cdot\delta X_i$$

$$\therefore \frac{\delta l^2 - \delta L^2}{\delta L^2} = 2r_{kp}\frac{\delta X_k}{\delta L}\cdot\frac{\delta X_p}{\delta L} = 2r_{ij}\frac{\delta X_i}{\delta L}\cdot\frac{\delta X_j}{\delta L} = 2r_{ij}M_iM_j \quad \text{........... (19)}$$

and $$\frac{dl^2 - dL^2}{dL^2} = 2r_{kp}\frac{dX_k}{dL}\cdot\frac{dX_p}{dL} = 2r_{ij}\frac{dX_i}{dL}\cdot\frac{dX_j}{dL} = 2r_{ij}N_iN_j$$

Again, $dx_i\delta x_i - dX_i\delta X_i = dx_k\delta x_k - dX_i\delta X_i$

$$= \frac{\partial x_k}{\partial X_i}\cdot\frac{\partial x_k}{\partial X_j}dX_i\delta X_j - dX_i\delta X_i$$

$$= \left(\frac{\partial x_k}{\partial X_i}\cdot\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right)dX_i\delta X_j$$

$$= 2r_{ij}\, dX_i\delta X_j \quad \text{......... (20)}$$

[where r_{ij} is given by (15)]

$$\therefore \frac{dx_i}{dL}\cdot\frac{\delta x_i}{\delta L} - \frac{dX_i}{dL}\cdot\frac{\delta X_i}{\delta L} = 2r_{ij}\frac{dX_i}{dL}\cdot\frac{\delta X_j}{\delta L}$$

or, $$\frac{dx_i}{dl}\cdot\frac{\delta x_i}{\delta l}\cdot\frac{dl}{dL}\cdot\frac{\delta l}{\delta L}-\frac{dX_i}{dL}\cdot\frac{\delta X_j}{\delta L}=2r_{ij}\frac{dX_i}{dL}\cdot\frac{\delta X_j}{\delta L}$$

or, $$\frac{dl}{dL}\cdot\frac{\delta l}{\delta L}\cos\theta-\cos\Theta=2r_{ij}N_iM_j \text{ .... (21)}$$

(using 16, 17, 18)

Equations (10), (19) and (21) shows that if $r_{ij}=0$, then $dl=dL$, $\delta l=\delta L, \theta=\Theta$. Thus when $r_{ij}=0$, then length of a line element and angle between two line elements remains unchanged during deformation and the body has undergone only rigid body deformation.

The necessary and sufficient condition for rigid body deformation at each point is $r_{ij}=0$. In other words r_{ij} cause a change in the length and oriettation of a line element when body is deformed. A nonzero tensor r_{ij} represents strain deformation and r_{ij} is known as ***strain tensor***, a finite strain tensor.

Note -1.

From (10) i.e., $\frac{dl^2-dL^2}{dL^2}=2r_{ij}N_iN_j$, we observe that $2r_{ij}N_iN_j$ is a scalar. But product N_iN_j of two vector components is known to be a tensor of order two. Therefore, by quotient law of tensor r_{ij} is a second order tensor known as Lagrngian Finite Strain Tensor.

Note -2.

From (15), i.e., $r_{ij}=\frac{1}{2}\left[\frac{\partial u_i}{\partial X_j}+\frac{\partial u_j}{\partial X_i}+\frac{\partial u_k}{\partial X_i}\cdot\frac{\partial u_k}{\partial X_j}\right]$, we observe that $r_{ij}=r_{ji}$. Therefore, tensor r_{ij} is symmetric.

1.17 Eulerian Finite Strain Tensor : Spatial Method

In Eulerian method i.e., spatial method of description, x_1, x_2, x_3, t are regarded as independent variables and the equation characterising the deformation as

$$X_k=X_k(x_1,x_2,x_3,t) \text{ .... (22)}$$

$$\therefore dX_k=\frac{\partial X_k}{\partial x_i}dx_i=\frac{\partial X_k}{\partial x_j}dx_j \text{ .... (23)}$$

Then $dL^2=dX_kdX_k=\frac{\partial X_k}{\partial x_i}\cdot\frac{\partial X_k}{\partial x_j}dx_idx_j$

and $dl^2=\delta_{ij}dx_idx_j$ .... (24)

$$\therefore \frac{dl^2 - dL^2}{dl^2} = 2\eta_{ij} n_i n_j \quad \text{.......... (25)}$$

where $$\eta_{ij} = \frac{1}{2}\left[\delta_{ij} - \frac{\partial X_k}{\partial x_i} \cdot \frac{\partial X_k}{\partial x_j}\right] \quad \text{........... (26)}$$

Now $dX_k = dx_k - du_k$

$$\therefore \frac{\partial X_k}{\partial x_i} = \delta_{ki} - \frac{\partial u_k}{\partial x_i}$$

and $$\frac{\partial X_k}{\partial x_j} = \delta_{kj} - \frac{\partial u_k}{\partial x_j}$$

$$\therefore \eta_{ij} = \frac{1}{2}\left[\delta_{ij} - \left(\delta_{ki} - \frac{\partial u_k}{\partial x_i}\right)\left(\delta_{kj} - \frac{\partial u_k}{\partial x_j}\right)\right]$$

$$= \frac{1}{2}\left[\delta_{ij} - \delta_{ki}\delta_{kj} + \frac{\partial u_k}{\partial x_i}\delta_{kj} + \frac{\partial u_k}{\partial x_j}\delta_{ki} - \frac{\partial u_k}{\partial x_i} \cdot \frac{\partial u_k}{\partial x_j}\right]$$

$$= \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i} \cdot \frac{\partial u_k}{\partial x_j}\right] \quad \text{..........(27)}$$

1.18 Change in the Angle :

Now, $dx_i \delta x_i - dX_i \delta x_i = \delta_{ij} dx_i \delta x_j - dX_k \delta X_k$

$$= \delta_{ij} dx_i \delta x_j - \frac{\partial X_k}{\partial x_i} \cdot \frac{\partial X_k}{\partial x_j} dx_i \delta x_j$$

$$\therefore \frac{dx_i}{dl} \cdot \frac{\delta x_i}{\delta l} - \frac{dX_i}{dL} \cdot \frac{\delta X_j}{\delta L} \cdot \frac{dL}{dl} \cdot \frac{\delta L}{\delta l} = 2\eta_{ij} \frac{dx_i}{dl} \cdot \frac{\delta x_j}{\delta l} = 2\eta_{ij} n_i m_j$$

or, $$\cos\theta - \frac{dL}{dl} \cdot \frac{\delta L}{\delta l} \cdot \cos\Theta = 2\eta_{ij} n_i m_j \quad \text{.......... (28)}$$

Now, if $\eta_{ij} = 0$, then $dL = dl, \delta L = \delta l, \theta = \Theta$. This corresponds to rigid body deformation. Hence, *the necessary and sufficient condition for rigid deformation at each point is* $\eta_{ij} = 0$. The nonzero tensor η_{ij} gives rise a strain deformation, and it can be regarded as the measure of this strain deformation. Resoning as before, from (27) we see that η_{ij} is a symmetric tensor of order two. It is known as Eulerian finite strain tensor.

1.19 Small Deformation : Infinitesimal Strain Tensor :

There are many common materials like metals, concrete, wood experience only small changes of shape when forces of reasonable magnitude are applied to them.

If the displacement gradients i.e., $\frac{\partial u_i}{\partial X_j}$ or $\frac{\partial u_i}{\partial x_j}$ are so small that squares and products of partial derivatives of u_i are negligible, then Lagrangian finite strain, tensor reduces to infinitesimal strain tensor.

Let E_{ij} and e_{ij} be the Lagrangian and Eulerian linear strain tensors respectively defined by

$$E_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j}(X_1, X_2, X_3) + \frac{\partial u_j}{\partial X_i}(X_1, X_2, X_3)\right] \text{..........(29)}$$

and $$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}(x_1, x_2, x_3) + \frac{\partial u_j}{\partial X_i}(x_1, x_2, x_3)\right] \text{........... (30)}$$

We first assume that all displacement gradients are numerically small compared to unity, so that

$$\left|\frac{\partial u_i}{\partial X_j}\right| << 1, (i, j = 1, 2, 3) \text{ .... (31)}$$

and we can neglect the squares and products of these quantities because they are small quantities of second order. So, the first order in the displacement gradients:

$$r_{ij}(X_1, X_2, X_3) = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i} \cdot \frac{\partial u_k}{\partial X_j}\right]$$

$$\cong \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right] = E_{ij}(X_1, X_2, X_3)$$

$$\therefore r_{ij} \cong E_{ij} \text{ ....(32)}$$

Also, we have $\frac{dl^2 - dL^2}{dL^2} \cong 2E_{ij}N_iN_j$ .... (33)

and $\frac{dl}{dL} \cdot \frac{\delta l}{\delta L} \cos\theta - \cos\Theta \cong 2E_{ij}N_iM_j$ .... (34)

Similarly, for $\left|\frac{\partial u_i}{\partial x_j}\right| << 1 (i, j = 1, 2, 3)$ .... (35)

we have $\eta_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}(x_1,x_2,x_3) + \frac{\partial u_j}{\partial x_i}(x_1,x_2,x_3) - \frac{\partial u_k}{\partial x_i}\cdot\frac{\partial u_k}{\partial x_j}\right]$

$$\cong \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] = e_{ij}$$

$$\therefore \eta_{ij} \cong e_{ij} \quad \text{......... (36)}$$

Also, $\frac{dl^2 - dL^2}{dl^2} \cong 2e_{ij}n_in_j$ .... (37)

and $\cos\theta - \frac{dL}{dl}\cdot\frac{\delta L}{\delta l}\cos\Theta \cong 2e_{ij}n_im_j$ .... (38)

Secondly we assume that displacement themselves are small in addition to displacement gradients so that we can neglect the product terms like $u_i\frac{\partial u_j}{\partial X_i}$.

Since,

$$x_i = X_i + u_i$$

$$\therefore u_i(x_1,x_2,x_3) = u_i(X_1+u_1, X_2+u_2, X_3+u_3)$$

$$= u_i(X_1,X_2,X_3) + u_j\frac{\partial u_i}{\partial X_j} + \text{......(by Taylor Expansion)}$$

$$\cong u_i(X_1,X_2,X_3)$$

Hence,

$$\frac{\partial u_i}{\partial X_j}(X_1,X_2,X_3) \cong \frac{\partial u_i}{\partial X_j}(x_1,x_2,x_3) = \frac{\partial u_i}{\partial x_k}(x_1,x_2,x_3)\cdot\frac{\partial x_k}{\partial X_j}$$

$$= \frac{\partial u_i}{\partial x_k}(x_1,x_2,x_3)\left[\delta_{kj} + \frac{\partial u_k}{\partial X_j}\right]$$

$$\cong \frac{\partial u_i}{\partial x_k}(x_1,x_2,x_3)\,\delta_{kj} \quad \text{[Neglecting product terms]}$$

$$= \frac{\partial u_i}{\partial x_j}(x_1,x_2,x_3) \quad \text{........., (39)}$$

Similarly, $\dfrac{\partial u_j}{\partial X_i}(X_1,X_2,X_3) \cong \dfrac{\partial u_j}{\partial x_i}(x_1,x_2,x_3)$

Therefore, we have

$$E_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j}(X_1,X_2,X_3) + \frac{\partial u_j}{\partial X_i}(X_1,X_2,X_3)\right]$$

$$\cong \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}(x_1,x_2,x_3) + \frac{\partial u_j}{\partial x_i}(x_1,x_2,x_3)\right]$$

$$= e_{ij} \;......... (40)$$

$$= \frac{1}{2}\left[u_{i,j} + u_{j,i}\right] \text{ (in comma notation)}$$

Thus if both the displacements and displacement gradients are small compared to one such that their squares and products can be neglected, Lagrangian linear strain tensor E_{ij} are identical, component by component to their counterparts in Eulerian linear strain tensor e_{ij}. Therefore we do not require any distinction between the two methods of description for the infinitesimal strain tensors, and simply call it infinitesimal or small strain tensor.

1.20 Geometrical Interpretation of Small Strain Components :

To give a geometrical interpretation of strains E_{11}, E_{22}, E_{33} we first consider the change in length of a material line element.

a) Extension of a material line element : Consider a material line element P_0Q_0 of length dL at $P_0(X_1,X_2,X_3)$ oriented in the direction of (N_1,N_2,N_3) in the undeformed body. After deformation it has take place PQ of length dl at $P(x_1,x_2,x_3)$ in the deformed body in the direction (n_1,n_2,n_3). Then

$$\frac{dl^2 - dL^2}{dL^2} = 2E_{ij}N_iN_j \;......... (41)$$

where $E_{ij} = \dfrac{1}{2}\left[\dfrac{\partial u_i}{\partial X_j} + \dfrac{\partial u_j}{\partial X_i}\right]$, being infinitesimal strain tensor

at P_0. From (41), we have

$$\frac{dl^2}{dL^2} = 1 + 2E_{ij}N_iN_j$$

or, $\dfrac{dl}{dL} = \left[1 + 2E_{ij}N_iN_j\right]^{1/2} = 1 + \dfrac{1}{2}\cdot 2E_{ij}N_iN_j + ...$

If the strain components are so small that we can neglect squares and products of E_{ij}, then we have

$$\frac{dl}{dL} = 1 + E_{ij}N_iN_j$$

or, $\dfrac{dl - dL}{dL} = E_{ij}N_iN_j$ .... (42)

Now L.H.S. of (42) is the extension per unit length of a line element oriented in the direction (N_1, N_2, N_3) and is called small extensional strain, and is denoted by $E_{(N)}$. Hence the small extensional strain is

$$E_{(N)} = E_{ij}N_iN_j \quad ....(43)$$

Geometrical Interpretation of E_{11}, E_{22}, E_{33}

If the line element was initially parallel to X_1 axis, then we have $N_1 = 1, N_2 = 0, N_3 = 0$.

$$\therefore E_{(1)} = E_{11}.$$

Thus E_{11} is the extension per unit original length of a line element which is initially parallel to X_1 axis. Similarly E_{22}, E_{33} represent, respectively, the extension of a line element per unit original length which are initially parallel to X_2 and X_3 axes. Here E_{11}, E_{22}, E_{33} are called extensional strain or normal strain.

To give geometrical interpretation of E_{23}, E_{31}, E_{12} we consider the change in angle between orthogonal line elements.

(b) Shear (or change in angle) between two orthogonal line elements:

If the material line P_0Q_0 and P_0R_0 of lengths dL and δL are orthogonal to each other in the undeformed state, i.e. $\Theta = \dfrac{\pi}{2}$, then from (34) we get,

$$\frac{dl}{dL}\cdot\frac{\delta l}{\delta L}\cos\theta - \cos\frac{\pi}{2} = 2E_{ij}N_iM_j$$

where $E_{ij} = \dfrac{1}{2}\left[\dfrac{\partial u_i}{\partial X_j} + \dfrac{\partial u_j}{\partial X_i}\right]$ = infinitesimal strain tensor at P_0.

$$\therefore \frac{dl}{dL}\cdot\frac{\delta l}{\delta L}\sin\left(\frac{\pi}{2} - \theta\right) = 2E_{ij}N_iM_j$$

or, $\sin\left(\frac{\pi}{2}-\theta\right)=\frac{2E_{ij}N_iM_j}{\frac{dl}{dL}\cdot\frac{\delta l}{\delta L}}$ .... (45)

Now, $\frac{\pi}{2}-\theta$ is the decrease in right angle between two orthogonal lines P_0Q_0 and P_0R_0 in the undeformed state and is called shear along the two lines. If $\gamma_{(NM)}$ be this shear along two orthogonal line elements initially oriented in the direction (N_1, N_2, N_3) and (M_1, M_2, M_3), then

$$\gamma_{(NM)}=\frac{\pi}{2}-\theta \text{ and hence}$$

$$\sin\gamma_{(NM)}=\frac{2E_{ij}N_iM_j}{\frac{dl}{dL}\cdot\frac{\delta l}{\delta L}} \quad \text{..........(46)}$$

If E_1 and E_2 are the extensions of P_0Q_0 and P_0R_0, respectively, then we get

$$\left.\begin{aligned} E_1&=\frac{dl-dL}{dL}\ i.e., \frac{dl}{dL}=1+E_1 \\ \text{and}\quad E_2&=\frac{\delta l-\delta L}{\delta L}\ i.e., \frac{\delta l}{\delta L}=1+E_2 \end{aligned}\right\} \quad \text{........... (47)}$$

Using (47) in (46) then we get

$$\sin\gamma_{(NM)}=\frac{2E_{ij}N_iM_j}{(1+E_1)(1+E_2)} \quad \text{........... (48)}$$

Hence, for small deformation, $\sin\gamma_{(NM)}=\gamma_{(NM)}$ and then

$$\gamma_{(NM)}=\frac{2E_{ij}N_iM_j}{1+E_1+E_2+E_1E_2}=2E_{ij}N_iM_j(1+E_1+E_2+E_1E_2)^{-1}$$

$$\cong 2E_{ij}N_iM_j \text{ [Neglecting small quantities] .... (49)}$$

Geometrical Interpretation of E_{12}, E_{23}, E_{31}:

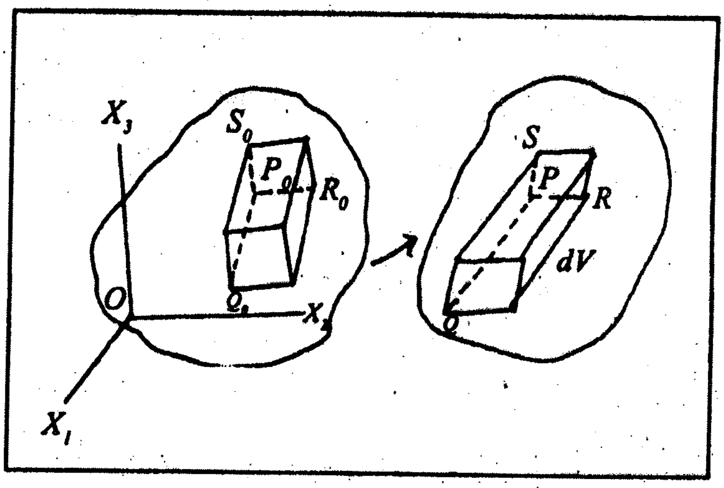
If we choose a pair of orthogonal line elements initially parallel to X_2 and X_3 axes, respectively, then we get

$$N_1=0, N_2=1, N_3=0; M_1=0, M_2=0, M_3=1; \gamma_{(23)}=2E_{23}$$

$$\therefore E_{23}=\frac{1}{2}\gamma_{(23)}.$$

Thus E_{23} represent half of the shear between two line elements which are initially paralle to X_2 and X_3 axes. Similar interpretation can be made in regard to E_{31} and E_{12}. Here E_{23}, E_{31}, E_{12} are called *shearing strains.* E_{ij} denote increase in length of a line element per unit original length or decrease in right angle between two lines elements. Thus for rigid deformation $E_{ij} = 0$.

1.21 Volumetric Strain or Cubical Dilatation :



Let us consider and elementary rectangular parallelopiped $P_0Q_0R_0S_0$ at $P_0(X_1, X_2, X_3)$ of volume element dV_0 at time $t = 0$ i.e., in the undeformed state, having edges of lengths $dX_1(=P_0Q_0), dX_2(=P_0R_0), dX_3(=P_0S_0)$, parallel to co-ordinate axes. So, in undeformed state,

$$dV_o = dX_1\, dX_2\, dX_3 \text{ ....} (50)$$

Due to small deformation, material points P_0, Q_0, R_0, S_0 moves to the points P, Q, R, S whose position vectors are $\bar{x}, \bar{x} + d\bar{x}^{(1)}, \bar{x} + d\bar{x}^{(2)}, \bar{x} + d\bar{x}^{(3)}$ so that rectangular parallelopiped $P_0Q_0R_0S_0$ deforms into a skewed parallelopiped $PQRS$ of volume dV in deformed state at time t.

$$\therefore dV = \left[d\bar{x}^{(1)}, d\bar{x}^{(2)}, d\bar{x}^{(3)}\right] = d\bar{x}^{(1)} \cdot d\bar{x}^{(2)} \times d\bar{x}^{(3)} \text{ ....} (51)$$

If $x_i, x_i + dx_i^{(1)}, x_i + dx_i^{(2)}, x_i + dx_i^{(3)}$ be the co-ordinates of P, Q, R, S respectively, then

$$dV = e_{ijk}\, dx_i^{(1)} dx_i^{(2)} dx_i^{(3)} \text{ ....} (52)$$

where e_{ijk} is alternating symbol defined as

$$e_{ijk} = \begin{cases} 0, \textit{if any two of } i, j, k \textit{ are equal}, \\ 1, \textit{if } i, j, k \textit{ are even permutation of } 1,2,3 \\ -1, \textit{if } i, j, k \textit{ are odd permutation of } 1,2,3 \end{cases} \text{ ....} (53)$$

In material i.e., Lagrangian description of deformation

$$x_i = x_i(X_1, X_2, X_3)$$

$$\therefore dx_i = \frac{\partial x_i}{\partial X_j} dX_j$$

So, $dx_i^{(1)} = \frac{\partial x_i}{\partial X_1} dX_1, dx_i^{(2)} = \frac{\partial x_i}{\partial X_2} dX_2, dx_i^{(3)} = \frac{\partial x_i}{\partial X_3} dX_3$ .... (54)

Using (54) in (52), we get

$$dV = e_{ijk} \frac{\partial x_i}{\partial X_1} \cdot \frac{\partial x_j}{\partial X_2} \cdot \frac{\partial x_k}{\partial X_3} dX_1\, dX_2\, dX_3$$

$$= J\, dV_0$$

where $J = e_{ijk} \frac{\partial x_i}{\partial X_1} \cdot \frac{\partial x_j}{\partial X_2} \cdot \frac{\partial x_k}{\partial X_3} = \left| \frac{\partial x_i}{\partial X_j} \right| = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_2}{\partial X_1} & \frac{\partial x_3}{\partial X_1} \\ \frac{\partial x_1}{\partial X_2} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_3}{\partial X_2} \\ \frac{\partial x_1}{\partial X_3} & \frac{\partial x_2}{\partial X_3} & \frac{\partial x_3}{\partial X_3} \end{vmatrix}$

$$= \frac{\partial(x_1; x_2, x_3)}{\partial(X_1, X_2, X_3)} \text{ .... (55)}$$

Now $J = \left| \frac{\partial x_i}{\partial X_j} \right| = \left| \frac{\partial (X_i + u_i)}{\partial X_j} \right| = \left| \delta_{ij} + u_{i,j} \right| = \left| \delta_{ij} + \frac{\partial u_i}{\partial X_j} \right|$.

$$\cong 1 + \frac{\partial u_i}{\partial x_i}, \text{ for small strain,}$$

$$= 1 + E_{11} + E_{22} + E_{33}$$

$$\therefore E_{11} + E_{22} + E_{33} = J - 1 = \frac{dV}{dV_0} - 1 = \frac{dV - dV_0}{dV_0} \text{ .... (56)}$$

Thus, for small strain $E_{11} + E_{22} + E_{33}$ geometrically represents the change in volume per unit original volume and is called *dilatation* or *volumetric strain*. Hence volumetric strain or cubical dilatation is equal to the sum of three linear strains.

1.22 Uniform Dilatation : When the strain quadric is a sphere, the principal axes of strains are indeterminate, and the extension (or contraction) of all linear elements issuing from a point is the same or we have

$$E_{11} = E_{22} = E_{33} = \frac{1}{3}\left(\frac{dV - dV_0}{dV_0} \right), E_{23} = E_{31} = E_{12} = 0.$$

1.23 Unit Summary : Here we have discussed simple and straight-forward method regarding the topic Strain.

1.24 Worked Out Examples

Ex.-1 *Given the displacement field :* $x_1 = X_1 + 2X_3, x_2 = X_2 - 2X_3, x_3 = X_3 - 2X_1 + 2X_2$, *determine the Lagrangian and Eulerian finite strain tensor.*

Ans. The displacements in the Lagrangian (material) method are

$$u_1 = x_1 - X_1 = 2X_3, u_2 = x_2 - X_2 = -2X_3$$

and $u_3 = x_3 - X_3 = -2X_1 + 2X_2$.

Since Lagrangian strain tensor is $r_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i} \cdot \frac{\partial u_k}{\partial X_j}\right]$

$$\therefore r_{11} = \frac{1}{2}\left[2\frac{\partial u_1}{\partial X_1} + \frac{\partial u_k}{\partial X_1} \cdot \frac{\partial u_k}{\partial X_1}\right] = \frac{1}{2}\left[2 \cdot 0 + 0 + 0 + (-2)^2\right] = 2,$$

$$r_{22} = \frac{1}{2}\left[2\frac{\partial u_2}{\partial X_2} + \frac{\partial u_k}{\partial X_2} \cdot \frac{\partial u_k}{\partial X_2}\right] = \frac{1}{2}\left[2 \cdot 0 + 0 + 0 + (-2)^2\right] = 2,$$

$$r_{33} = \frac{1}{2}\left[2\frac{\partial u_3}{\partial X_3} + \frac{\partial u_k}{\partial X_3} \cdot \frac{\partial u_k}{\partial X_3}\right] = \frac{1}{2}\left[2 \cdot 0 + 2^2 + (-2)^2 + 0\right] = 4,$$

$$r_{21} = r_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_k}{\partial X_1} \cdot \frac{\partial u_k}{\partial X_2}\right] = \frac{1}{2}[0 + 0 + 0 + 0 + (-2)(2)] = -2,$$

$$r_{31} = r_{13} = \frac{1}{2}\left[\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} + \frac{\partial u_k}{\partial X_1} \cdot \frac{\partial u_k}{\partial X_3}\right] = \frac{1}{2}[2 - 2 + 0 + 0 + (-2) \cdot 0] = 0,$$

$$r_{32} = r_{23} = \frac{1}{2}\left[\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_k}{\partial X_2} \cdot \frac{\partial u_k}{\partial X_3}\right] = \frac{1}{2}[-2 + 2 + 0 + 0 + 0] = 0.$$

Hence, $(r_{ij}) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

For Eulerian (spatial) method, at first we express X_1, X_2, X_3 in terms of x_1, x_2, x_3. Now from first two given equations we get

$$x_1 + x_2 = X_1 + X_2$$

and from second and third given equations we get

$$x_2 + 2x_3 = -4X_1 + 5X_2$$

$$\therefore X_2 = \frac{4x_1 + 5x_2 + 2x_3}{9}$$

and $X_1 = \dfrac{5x_1 + 4x_2 - 2x_3}{9}$.

Also, $X_3 = x_3 + 2X_1 - 2X_2 = \dfrac{2x_1 - 2x_2 + x_3}{9}$.

Hence, the displacements in Eulerian description, we have

$$u_1 = x_1 - X_1 = (4x_1 - 4x_2 + 2x_3)/9,$$

$$u_2 = x_2 - X_2 = (4x_2 - 4x_1 + 2x_3)/9,$$

$$u_3 = x_3 - X_3 = (8x_3 - 2x_1 + 2x_2)/9.$$

Now Eulerian strain tensor is $\eta_{ij} = \dfrac{1}{2}\left[\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} - \dfrac{\partial u_k}{\partial x_i}\cdot\dfrac{\partial u_k}{\partial x_j}\right]$.

$$\therefore \eta_{11} = \frac{1}{2}\left[2\frac{\partial u_1}{\partial x_1} - \frac{\partial u_k}{\partial x_1}\cdot\frac{\partial u_k}{\partial x_1}\right] = \frac{1}{2}\left[2\cdot\frac{4}{9} - \left(\frac{4}{9}\right)^2 - \left(-\frac{4}{9}\right)^2 - \left(-\frac{2}{9}\right)^2\right] = \frac{2}{9},$$

$$\eta_{22} = \frac{1}{2}\left[2\frac{\partial u_2}{\partial x_2} - \frac{\partial u_k}{\partial x_2}\cdot\frac{\partial u_k}{\partial x_2}\right] = \frac{1}{2}\left[2\cdot\frac{4}{9} - \left(-\frac{4}{9}\right)^2 - \left(\frac{4}{9}\right)^2 - \left(\frac{2}{9}\right)^2\right] = \frac{2}{9},$$

$$\eta_{33} = \frac{1}{2}\left[2\frac{\partial u_3}{\partial x_3} - \frac{\partial u_k}{\partial x_3}\cdot\frac{\partial u_k}{\partial x_3}\right] = \frac{1}{2}\left[2\cdot\frac{8}{9} - \left(-\frac{2}{9}\right)^2 - \left(-\frac{2}{9}\right)^2 - \left(\frac{8}{9}\right)^2\right] = \frac{4}{9},$$

$$\eta_{12} = \eta_{21} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} - \frac{\partial u_k}{\partial x_1}\cdot\frac{\partial u_k}{\partial x_2}\right] = \frac{1}{2}\left[\left(-\frac{4}{9}\right) + \left(-\frac{4}{9}\right) - \frac{4}{9}\cdot\left(-\frac{4}{9}\right) - \left(-\frac{4}{9}\right)\left(\frac{4}{9}\right) - \left(-\frac{2}{9}\right)\left(\frac{2}{9}\right)\right] = -\frac{2}{9},$$

$$\eta_{23} = \eta_{32} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} - \frac{\partial u_k}{\partial x_2}\cdot\frac{\partial u_k}{\partial x_3}\right] = \frac{1}{2}\left[-\frac{2}{9} + \left(+\frac{2}{9}\right) - \left(-\frac{4}{9}\right)\left(\frac{2}{9}\right) - \frac{4}{9}\cdot\left(-\frac{2}{9}\right) - \frac{2}{9}\cdot\frac{8}{9}\right] = 0.$$

$$\eta_{31} = \eta_{13} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} - \frac{\partial u_k}{\partial x_1}\cdot\frac{\partial u_k}{\partial x_3}\right] = \frac{1}{2}\left[\frac{2}{9} + \left(-\frac{2}{9}\right) - \frac{4}{9}\cdot\frac{2}{9} - \left(-\frac{4}{9}\right)\cdot\left(-\frac{2}{9}\right) - \left(-\frac{2}{9}\right)\cdot\frac{8}{9}\right] = 0.$$

Hence, Eulerian finite strain tensor is

$$(\eta_{ij})=\frac{1}{9}\begin{pmatrix}2 & -2 & 0\\ -2 & 2 & 0\\ 0 & 0 & 4\end{pmatrix}.$$

Ex-2. *If the equations characterizing the deformation are given by*

$x_1 = X_1 + \in X_2, x_2 = X_2 - \in X_1 + \in X_3, x_3 = X_3 - \in X_2$ *determine the Lagrangian and Eulerian finite strain tensors. Also find infinite simal strain tensor when* $\in$ *is a small parameter.*

Ans. The displacements in Lagrangian methods are

$$u_1 = x_1 - X_1 = \in X_2, u_2 = x_2 - X_2 = -\in X_1 + \in X_3, u_3 = x_3 - X_3 = -\in X_2.$$

$$\therefore r_{11} = \frac{1}{2}\left[2\frac{\partial u_1}{\partial X_1} + \frac{\partial u_k}{\partial X_1}\cdot\frac{\partial u_k}{\partial X_1}\right] = \frac{1}{2}\left[2.0 + 0 + (-\in)^2 + 0\right] = \frac{\in^2}{2},$$

$$r_{22} = \frac{1}{2}\left[2\frac{\partial u_2}{\partial X_2} + \frac{\partial u_k}{\partial X_2}\cdot\frac{\partial u_k}{\partial X_2}\right] = \frac{1}{2}\left[2.0 + (\in)^2 + 0 + (-\in)^2\right] = \in^2,$$

$$r_{33} = \frac{1}{2}\left[2\frac{\partial u_3}{\partial X_3} + \frac{\partial u_k}{\partial X_3}\cdot\frac{\partial u_k}{\partial X_3}\right] = \frac{1}{2}\left[2.0 + 0 + (\in)^2 + 0\right] = \frac{\in^2}{2},$$

$$r_{12} = r_{21} = \frac{1}{2}\left[\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_k}{\partial X_1}\cdot\frac{\partial u_k}{\partial X_2}\right] = \frac{1}{2}\left[\in - \in + 0 + 0 + 0\right] = 0,$$

$$r_{23} = r_{32} = \frac{1}{2}\left[\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_k}{\partial X_2}\cdot\frac{\partial u_k}{\partial X_3}\right] = \frac{1}{2}\left[\in - \in + 0 + 0 + 0\right] = 0,$$

$$r_{31} = r_{13} = \frac{1}{2}\left[\frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_3} + \frac{\partial u_k}{\partial X_1}\cdot\frac{\partial u_k}{\partial X_3}\right] = \frac{1}{2}\left[0 + 0 + 0 + (-\in)\in + 0\right] = -\frac{\in^2}{2}$$

$$\text{Hence, } (r_{ij}) = \begin{pmatrix}\frac{\in^2}{2} & 0 & -\frac{\in^2}{2}\\ 0 & \in^2 & 0\\ -\frac{\in^2}{2} & 0 & \frac{\in^2}{2}\end{pmatrix}.$$

From given equations we have $X_2 = \dfrac{\in x_1 + x_2 - \in x_3}{1+2\in^2}$, $X_1 = x_1 - \dfrac{\in^2 x_1 + \in x_2 - \in^2 x_3}{1+2\in^2}$,

and $X_3 = x_3 + \dfrac{\in^2 x_1 + \in x_2 - \in^2 x_3}{1+2\in^2}$.

$$\therefore \quad u_1 = x_1 - X_1 = \in(\in x_1 + x_2 - \in x_3)/(1+2\in^2),$$

$$u_2 = x_2 - X_2 = (-\in x_1 + 2\in^2 x_2 + \in x_3)/(1+2\in^2),$$

$$u_3 = x_3 - X_3 = -\in(\in x_1 + x_2 - \in x_3)/(1+2\in^2).$$

So, $\eta_{11} = \dfrac{1}{2}\left[2\dfrac{\partial u_1}{\partial x_1} - \dfrac{\partial u_k}{\partial x_1}\cdot\dfrac{\partial u_k}{\partial x_1}\right] = \dfrac{1}{2}\left[2\cdot\dfrac{\in^2}{1+2\in^2} - \left(\dfrac{\in^2}{1+2\in^2}\right)^2 - \left(\dfrac{-\in}{1+2\in^2}\right)^2 - \left(\dfrac{-\in^2}{1+2\in^2}\right)^2\right] = \dfrac{\in^2}{2(1+2\in^2)}$

$$\eta_{22} = \frac{1}{2}\left[2\frac{\partial u_2}{\partial x_2} - \frac{\partial u_k}{\partial x_2}\cdot\frac{\partial u_k}{\partial x_2}\right] = \in^2/(1+2\in^2),$$

$$\eta_{33} = \frac{1}{2}\left[2\frac{\partial u_3}{\partial x_3} - \frac{\partial u_k}{\partial x_3}\cdot\frac{\partial u_k}{\partial x_3}\right] = \in^2/2(1+2\in^2),$$

$$\eta_{12} = \eta_{21} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_k}{\partial x_1}\cdot\frac{\partial u_k}{\partial x_2}\right] = \frac{1}{2}\left[\frac{\in}{1+2\in^2} + \left(\frac{-\in}{1+2\in^2}\right) - \frac{\in^2}{1+2\in^2}\cdot\frac{\in}{1+2\in^2} - \left(\frac{-\in}{1+2\in^2}\right)\left(\frac{2\in^2}{1+2\in^2}\right) - \left(\frac{-\in^2}{1+2\in^2}\right)\left(\frac{-\in}{1+2\in^2}\right)\right] = 0,$$

$$\eta_{23} = \eta_{32} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} - \frac{\partial u_k}{\partial x_2}\cdot\frac{\partial u_k}{\partial x_3}\right] = \frac{1}{2}\left[\frac{\in}{1+2\in^2} + \left(\frac{-\in}{1+2\in^2}\right) - \frac{\in}{1+2\in^2}\left(\frac{-\in^2}{1+2\in^2}\right) - \frac{2\in^2}{1+2\in^2}\cdot\frac{\in}{1+2\in^2} - \left(\frac{-\in}{1+2\in^2}\right)\left(\frac{+\in^2}{1+2\in^2}\right)\right] = 0,$$

$$\eta_{31} = \eta_{13} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} - \frac{\partial u_k}{\partial x_1}\cdot\frac{\partial u_k}{\partial x_3}\right] = \frac{1}{2}\left[\frac{-\in^2}{1+2\in^2} - \frac{\in^2}{1+2\in^2} - \frac{\in^2}{1+2\in^2}\cdot\left(\frac{-\in^2}{1+2\in^2}\right) - \left(\frac{-\in}{1+2\in^2}\right)\left(\frac{\in}{1+2\in^2}\right) - \left(\frac{-\in^2}{1+2\in^2}\right)\left(\frac{\in^2}{1+2\in^2}\right)\right] = -\frac{\in^2}{2(1+2\in^2)}$$

Hence, $(\eta_{ij})=\frac{1}{1+2\epsilon^2}\begin{pmatrix}\frac{\epsilon^2}{2} & 0 & -\frac{\epsilon^2}{2}\\ 0 & \epsilon^2 & 0\\ -\frac{\epsilon^2}{2} & 0 & \frac{\epsilon^2}{2}\end{pmatrix}$.

Infinitesimal strain tensor is

$$E_{ij}=e_{ij}=\frac{1}{2}\left[\frac{\partial u_i}{\partial X_j}+\frac{\partial u_j}{\partial X_i}\right].$$

$$\therefore E_{11}=\frac{\partial u_1}{\partial X_1}=0, E_{22}=\frac{\partial u_2}{\partial X_2}=0, E_{33}=\frac{\partial u_3}{\partial X_3}=0,$$

$$E_{21}=E_{12}=\frac{1}{2}\left[\frac{\partial u_1}{\partial X_2}+\frac{\partial u_2}{\partial X_1}\right]=\frac{1}{2}[\epsilon+(-\epsilon)]=0,$$

$$E_{23}=E_{32}=\frac{1}{2}\left[\frac{\partial u_2}{\partial X_3}+\frac{\partial u_3}{\partial X_2}\right]=\frac{1}{2}[\epsilon-\epsilon]=0,$$

$$E_{31}=E_{13}=\frac{1}{2}\left[\frac{\partial u_1}{\partial X_3}+\frac{\partial u_3}{\partial X_1}\right]=\frac{1}{2}[0+0]=0.$$

$$\therefore (E_{ij})=(e_{ij})=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Ex. 3. *The strain tensor at a point is given by* $(E_{ij})=\begin{pmatrix}5 & 3 & 0\\ 3 & 4 & -1\\ 0 & -1 & 2\end{pmatrix}$. *Determine the extension of a line element in the direction of* $\left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$. *What is the change of angle between two perpendicular line elements in the directions of* $\left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$ *and* $\left(\frac{1}{\sqrt{5}},0,\frac{2}{\sqrt{5}}\right)$?

Ans. We know that the small extensional strain, $E_{(N)}=E_{ij}N_iN_j$.

$\therefore$ extension of a line element in the direction of $\left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$ is

$$E_{(N)} = E_{11}N_1N_1 + E_{22}N_2N_2 + E_{33}N_3N_3 + 2E_{12}N_1N_2 + 2E_{23}N_2N_3 + 2E_{31}N_3N_1$$

$$= 5\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right)^2 + 2\left(\frac{1}{3}\right)^2 + 2\cdot 3\cdot\frac{2}{3}\cdot\frac{2}{3} + 2(-1)\cdot\frac{2}{3}\cdot\frac{1}{3} + 0$$

$$= 5\cdot\frac{4}{9} + 4\cdot\frac{4}{9} + 2\cdot\frac{1}{9} + \frac{24}{9} - \frac{4}{9}$$

$$= \frac{58}{9}.$$

The change of angle between two perpendicular line elements in the directions of (N_1, N_2, N_3) and (M_1, M_2, M_3) is

$$\gamma_{(NM)} = 2E_{ij}N_iM_j = 2\big[E_{11}N_1M_1 + E_{22}N_2M_2 + E_{33}N_3M_3$$

$$E_{12}N_1M_2 + E_{21}N_2M_1 + E_{23}N_2M_3 + E_{32}N_3M_2 + E_{31}N_3M_1 + E_{13}N_1M_3\big]$$

$$= 2\left[\frac{10}{3\sqrt{5}} - \frac{4}{3\sqrt{5}} + 3\cdot\frac{2}{3\sqrt{5}} + \frac{4}{3\sqrt{5}}\right]$$

$$= \frac{32}{3\sqrt{5}}.$$

Ex-4. *If the equations characterizing the deformation are given by* $x_1 = X_1 + \in X_2, x_2 = X_2 + \in X_3, x_3 = X_3 + \in X_1$, *where* $\in$ *is small, determine the infinitesimal strain tensor.*

Ans. Here the displacements are

$$u_1 = x_1 - X_1 = \in X_2$$

$$u_2 = x_2 - X_2 = \in X_3$$

$$u_3 = x_3 - X_3 = \in X_1$$

$\therefore E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$ gives

$$E_{11} = \frac{\partial u_1}{\partial X_1} = 0, E_{22} = \frac{\partial u_2}{\partial X_2} = 0, E_{33} = \frac{\partial u_3}{\partial X_3} = 0,$$

$$E_{12} = E_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) = \frac{1}{2}(\in +0) = 0 = \frac{\in}{2},$$

$$E_{13} = E_{31} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) = \frac{1}{2}(0 + \in) = \frac{\in}{2},$$

$$E_{23} = E_{32} = \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) = \frac{1}{2}(\in +0) = \frac{\in}{2}.$$

$$\therefore (E_{ij}) = \begin{pmatrix} 0 & \frac{\in}{2} & \frac{\in}{2} \\ \frac{\in}{2} & 0 & \frac{\in}{2} \\ \frac{\in}{2} & \frac{\in}{2} & 0 \end{pmatrix}.$$

Again from given relations we get

$$X_1 = \left(x_1 - \in x_2 + \in^2 x_3\right)/\left(1+\in^3\right),$$

$$X_2 = \left(x_1 \in^2 + x_2 - \in x_3\right)/\left(1+\in^3\right),$$

$$X_3 = \left(-x_1 \in + \in^2 x_2 + x_3\right)/\left(1+\in^3\right).$$

$$\therefore \quad u_1 = x_1 - X_1 = x_1 - \frac{\left(x_1 - \in x_2 + \in^2 x_3\right)}{1+\in^3},$$

$$u_2 = x_2 - X_2 = x_2 - \frac{\left(x_1 \in^2 + x_2 - \in x_3\right)}{1+\in^3},$$

$$u_3 = x_3 - X_3 = x_3 - \frac{\left(-x_1 \in + \in^2 x_2 + x_3\right)}{1+\in^3}.$$

Therefore, $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ gives

$$e_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = 1 - \frac{1}{1+\in^3} = \frac{\in^3}{1+\in^3},$$

$$e_{22} = \frac{\partial u_2}{\partial x_2} = 1 - \frac{1}{1+\in^3} = \frac{\in^3}{1+\in^3},$$

$$e_{33} = \frac{\partial u_3}{\partial x_3} = 1 - \frac{1}{1+\epsilon^3} = \frac{\epsilon^3}{1+\epsilon^3},$$

$$e_{12} = e_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left[\frac{\epsilon}{1+\epsilon^3} + \left(\frac{-\epsilon^2}{1+\epsilon^3}\right)\right] = \frac{1}{2}\frac{\epsilon(1-\epsilon)}{1+\epsilon^3},$$

$$e_{23} = e_{32} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\left[\frac{\epsilon}{1+\epsilon^3} - \frac{\epsilon^2}{1+\epsilon^3}\right] = \frac{1}{2}\frac{\epsilon(1-\epsilon)}{1+\epsilon^3},$$

$$e_{31} = e_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\left[\frac{-\epsilon^2}{1+\epsilon^3} + \frac{\epsilon}{1+\epsilon^3}\right] = \frac{1}{2}\frac{\epsilon(1-\epsilon)}{1+\epsilon^3}.$$

$$\therefore (e_{ij}) = \frac{1}{1+\epsilon^3}\begin{pmatrix} \epsilon^3 & \frac{\epsilon(1-\epsilon)}{2} & \frac{\epsilon(1-\epsilon)}{2} \\ \frac{\epsilon(1-\epsilon)}{2} & \epsilon^3 & \frac{\epsilon(1-\epsilon)}{2} \\ \frac{\epsilon(1-\epsilon)}{2} & \frac{\epsilon(1-\epsilon)}{2} & \epsilon^3 \end{pmatrix}.$$

For small ϵ, we have, $\epsilon^3 \approx 0, \epsilon^2 \approx 0$. Therefore the infinitesimal strain tensor is

$$\therefore (E_{ij}) = (e_{ij}) = \begin{pmatrix} 0 & \frac{\epsilon}{2} & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & 0 & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & \frac{\epsilon}{2} & 0 \end{pmatrix}.$$

Ex.-5. e_{ij} are given as follows:

$e_{11} = 0, e_{22} = 0 = e_{33} = e_{13} = e_{12}, e_{23} = x_2 x_3$. Find u_i.

Ans. Given that

$$e_{11} = 0, \text{ i.e., } \frac{\partial u_1}{\partial x_1} = 0, \text{ .... (i)}$$

$$e_{22} = 0, \text{ i.e., } \frac{\partial u_2}{\partial x_2} = 0, \text{ .... (ii)}$$

$e_{33} = 0, i.e., \frac{\partial u_3}{\partial x_3} = 0,$ .... (iii)

$e_{12} = 0, i.e., \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0,$ .... (iv)

$e_{13} = 0, i.e., \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0,$ .... (v)

and $e_{23} = x_2 x_3, i.e., \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = x_2 x_3$ .... (vi)

Integrating (ii) and (iii), then we get

$u_2 = f(x_1, x_3)$ and $u_3 = g(x_1, x_2)$.......... (vii)

Using (vii) in (vi), then we get

$$\frac{\partial}{\partial x_3} \frac{f(x_1, x_3)}{} + \frac{\partial}{\partial x_2} \frac{g(x_1, x_2)}{} = x_2 \cdot x_3$$

or, $$\frac{\partial}{\partial x_3} \frac{f(x_1, x_3)}{} + \frac{\partial}{\partial x_2} \frac{g(x_1, x_2)}{} - x_2 x_3 = 0$$

Since the first and second term of the above equation can not have terms of the form $x_2 x_3$, so the equation can never be satisfied.

Thus for the given e_{ij}, the equation $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ do not yields solution for u_i.

1.25 Self Assessment Questions:

1. If the deformation of a body is defined by displacement components $u_1 = K(3X_1^2 + X_2)$, $u_2 = K(X_2^2 + X_3), u_3 = K(4X_3 + X_1)$ where K>0. Compute the extension of a line element that passes through the point (1,1,1) in the direction $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

2. If the equations characterizing the deformation are given by $x_1 = X_1 + \lambda X_2$, $x_2 = X_2 + \lambda X_3$, $x_3 = X_3 + \lambda X_2$, determine (E_{ij}) and (e_{ij}). Hence show that $(E_{ij}) = (e_{ij})$ for small λ.

3. A displacement field is given by $x_1 = X_1 - cX_2 + bX_3, x_2 = cX_1 + X_2 - aX_3, x_3 = -bX_1 + aX_2 + X_3$. Find (r_{ij}) and (η_{ij}).

4. For the displacement field $u_1 = (X_1 - X_3)^2, u_2 = (X_2 + X_3)^2, u_3 = -X_1X_2$. Compute the change in right angle between $\vec{N} = \frac{8\hat{e}_1 - \hat{e}_2 + 4\hat{e}_3}{9}$ and $\vec{M} = \frac{4\hat{e}_1 - 4\hat{e}_2 + 7\hat{e}_3}{9}$ at the point (0, 2, –1).

5. Consider the displacement components $u_1 = a_1x_1 + b_1x_2 + c_1x_3, u_2 = a_2x_1 + b_2x_2 + c_2x_3,$ $u_3 = a_3x_1 + b_3x_2 + c_3x_3$. Find infinitesimal strain tensor and finite strain tensor.

6. Find (r_{ij}) and (η_{ij}) for deformation $x_1 = a_1(X_1 + \alpha X_2), x_2 = a_2X_2, x_3 = a_3X_3$, where a_1, a_2, a_3 are constants.

1.26 Further Suggested Readings

1. Continuum Mechanics : T.J. Chung, Prentice-Hall.
2. Schaum's Outline of theory and problem of continuum mechanics. Gedrge R. Mase, McGraw-Hill.
3. Continuum Mechanics: A.J.M. Spencer, Longman.
4. Mathematical Theory of Continuum Mechanics : R.N. Chatterjee, Narosa Publishing House.
5. Foundation of Fluid Mechanics: S.W. Yuan, Prentice-Hall.
6. Fluid Dynamics : J.K. Goyal, K.P. Gupta, Pragati Prakashan.
7. Textbook of Fluid Dynamics : F.Chorlton, CBS Publishers and Distributors.
8. Theory of Elasticity : Yu. Amenzade, Mir Publishers, Moscow.
9. Applied Elasticity : C.T. Wang, McGraw-Hill.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-V : Group – A : Marks – 50

Module No. - 50
(Mechanics of Continuous Media)

STRUCTURE :

2.1 Introduction :

This Module is the continuation of the Module No. 49. Here we discuss the analysis part of the Strain for continuous media.

2.2 Objectives :

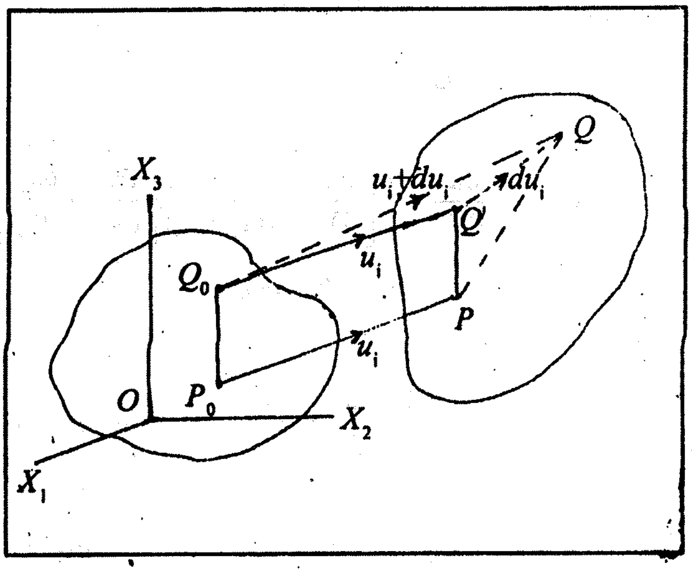
Without strain analysis, the concept of strain will not be fulfilled. This module will learn the analysis technique of strain analysis.

2.3 Key words :

Relative displacement, strain vector, rotation vector, strain quadric, principal strains, principal directions, strain invariants, Isochoric deformation, compatibility equations, deformation gradient tensor, Jocobian of the deformation.

2.4 Relative Displacement :

Let us consider the two neighbouring material points $P_0(X_1, X_2, X_3)$ and $Q_0(X_1 + dX_1 + dX_2 + X_3 + dX_3)$ in the undeformed state experience a displacement $\vec{u} = (u_1, u_2, u_3)$ and $\vec{u} + d\vec{u} = (u_1 + du_1, u_2 + du_2, u_3 + du_3)$ respectively, i.e., P_0 moves to P and Q_0 moves to Q in the deformed state. Here $\overline{P_0P} = \overline{Q_0Q^1} = \vec{u}$ and $\overline{Q_0Q} = \vec{u} + d\vec{u}$



$\therefore \overline{Q^1Q} = d\vec{u}$, is the relative displacement of material point at Q_0 w.r.t. material point at P_0. Hence for material method (Lagrangian) of description, u_i must be functions of X_1, X_2, X_3, i.e.,

$$u_i = F_i(X_1, X_2, X_3) \text{ ....} (57)$$

and hence $u_i + du_i$ will be similar functions of $X_1 + dX_1, X_2 + dX_2, X_3 + dX_3$.

So, $u_i + du_i = F_i(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3)$

$$= F_i(X_1, X_2, X_3) + \left(\frac{\partial F_i}{\partial X_1} \cdot dX_1 + \frac{\partial F_i}{\partial X_2} \cdot dX_2 + \frac{\partial F_i}{\partial X_3} dX_3\right) + \cdots\cdots$$

[by Taylor's series expansion]

Since the points P_0 and Q_0 are very close together, so dX_i must be small and we can neglect the terms containing powers of dX_i higher than first. So,we have

$$u_i + du_i = F_i(X_1, X_2, X_3) + \frac{\partial F_i}{\partial X_1} \cdot dX_1 + \frac{\partial F_i}{\partial X_2} \cdot dX_2 + \frac{\partial F_i}{\partial X_3} \cdot dX_3$$

$$= u_i + \frac{\partial F_i}{\partial X_j} dX_j$$

$$\therefore du_i = \frac{\partial u_i}{\partial X_j} \cdot dX_j \cdots\cdots\cdots (58)$$

Now, $\dfrac{\partial u_i}{\partial X_j} = \dfrac{1}{2}\left[\dfrac{\partial u_i}{\partial X_j} + \dfrac{\partial u_j}{\partial X_i}\right] + \dfrac{1}{2}\left[\dfrac{\partial u_i}{\partial X_j} - \dfrac{\partial u_j}{\partial X_i}\right]$

$$= E_{ij} + R_{ij}, \text{ say.................................} (59)$$

where $E_{ij} = \dfrac{1}{2}\left[\dfrac{\partial u_i}{\partial X_j} + \dfrac{\partial u_j}{\partial X_i}\right] = E_{ji}$, symmetric small strain tensor of order 2, and

$R_{ij} = \dfrac{1}{2}\left[\dfrac{\partial u_i}{\partial X_j} - \dfrac{\partial u_j}{\partial X_i}\right] = -R_{ji}$, skew-symmetric tensor of order 2.

Now from (58) and (59) we get

$$du_i = E_{ij} dX_j + R_{ij} dX_j \text{ ....} (60)$$

Again we form a vector R_i by setting

$$R_i = e_{ijk} R_{kj} \text{ ....} (61)$$

$$e_{ijk}R_i = e_{ijk}e_{ipq}R_{qp} = (\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp})R_{qp}$$

$$= R_{kj} - R_{jk}$$

$$= 2R_{kj} \left(\because R_{jk} = -R_{kj}\right) ....(62)$$

Hence, $R_{kj} = \frac{1}{2}e_{ijk}R_i$, which is the inverse relation of (61).

Also, $R_{ij}dX_j = \frac{1}{2}e_{kji}R_k dX_j$ (Using (62))

$$= \frac{1}{2}\left(\vec{R} \times d\vec{X}\right)_i ....(63)$$

Where, $\vec{R} = (R_1, R_2, R_3), d\vec{X} = \overline{P_0Q_0} = (dX_1, dX_2, dX_3)$

Now, $R_i = e_{ijk}R_{kj} = \frac{e_{ijk}}{2}\left(\frac{\partial u_k}{\partial X_j} - \frac{\partial u_j}{\partial X_k}\right) = \frac{1}{2}\left[e_{ijk}\frac{\partial u_k}{\partial X_j} - e_{ijk}\frac{\partial u_j}{\partial X_k}\right]$

$$= \frac{1}{2}\left[e_{ijk}\frac{\partial u_k}{\partial X_j} - e_{ikj}\frac{\partial u_k}{\partial X_j}\right] = \frac{1}{2}\left[e_{ijk}\frac{\partial u_k}{\partial X_j} + e_{ijk}\frac{\partial u_k}{\partial X_j}\right].$$

$$= e_{ijk}\frac{\partial u_k}{\partial X_i} = \left(rot\ \vec{u}\right)_i$$

$\therefore \vec{R} = rot\ \vec{u}$ ....(64)

Therefore the part $R_{ij}dX_j$ represents a relative displacement involving small rigid body rotation of the neighbourhood element of P_0 through an angle $\frac{1}{2}\vec{R} = \frac{1}{2}rot\ \vec{u}$. The vector $\vec{R}$ is called *small rotation vector* and R_{ij} is called *small rotation tensor.*

2.5 Strain Vector :

Suppose a line element P_0Q_0 at P_0 oriented in the direction (N_1, N_2, N_3) in the undeformed state of the continuum body deforms into a line element PQ at P in the deformed state, then the strain vector at P_0 is defined as

$$\bar{E}^{(N)} = \frac{\overline{PQ} - \overline{P_0Q_0}}{\left|\overline{P_0Q_0}\right|}$$

If (X_1, X_2, X_3) and $(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3)$ be the co-ordinates of P_0 and Q_0 in the undeformed state of the line element P_0Q_0 of length dL, then

$$N_i = \frac{dX_i}{dL}$$

Again, from $\bar{E}^{(N)}$ we have

$$\bar{E}^{(N)} = \frac{\overline{PQ} - \overline{P_0Q_0}}{\left|\overline{P_0Q_0}\right|} = \frac{\overline{PQ} - \overline{PQ'}}{dL} \quad (\because P_0Q_0 = PQ' \text{ and } P_0Q_0 \parallel PQ')$$

$$= \frac{\overline{Q'Q}}{dL} = \frac{\overline{Q_0Q} - \overline{Q_0Q'}}{dL}$$

$$\therefore E_i^{(N)} = \frac{(u_i + du_i) - u_i}{dL} = \frac{du_i}{dL} \text{ .... (65)}$$

If the deformation consists of strain deformation only involving no rigid body deformation, then relative displacement is given by

$$du_i = E_{ij}dX_j \text{ .... (66)}$$

where E_{ij} is the small strain tensor at P_0.

$$\therefore E_i^{(N)} = \frac{E_{ij}dX_j}{dL} = E_{ij}N_j \text{ .... (67)}$$

which gives the relation between strain vector and strain tensor at P_0.

Normal components of strain vector $\bar{E}^{(N)}$ in the direction (N_1, N_2, N_3) is given by

$$E_i^{(N)}N_i = E_{ij}N_jN_i = E_{(N)}$$

$$= E_{11} \text{ when } N_1 = 1, N_2 = 0 = N_3,$$

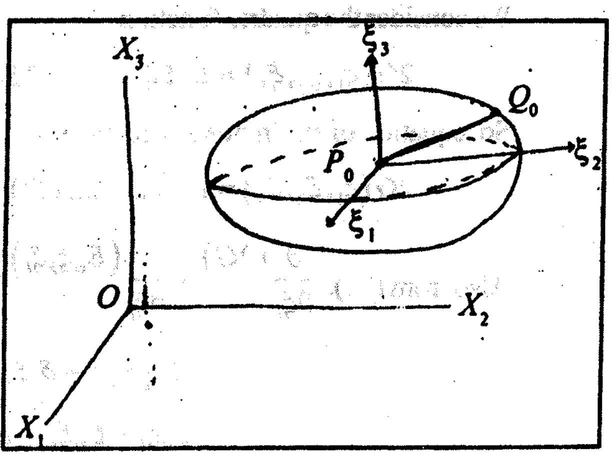
$$= E_{22} \text{ when } N_2 = 1, N_1 = 0 = N_3,$$

$$= E_{33} \text{ when } N_3 = 1, N_1 = 0 = N_2.$$

This is the reason extensional strain E_{11}, E_{22}, E_{33} are also called normal strain.

2.6 Strain Quadric :

The state of deformation in the neighbourhood of a point P_0 in undeformed state of a continuum body can be understood more clearly by a geometrical treatment.



Let $P_0(X_1, X_2, X_3)$ be a point in the undeformed state of the continuum body. The axes OX_1, OX_2, OX_3 are fixed in space. Let E_{ij} be the small strain tensor at P_0 w.r.t. this set of axes.

Now we introduce a local system of axes $P_0\xi_1, P_0\xi_2, P_0\xi_3$ with P_0 as origin and axes are parallel to OX_1, OX_2, OX_3 respectively. Then, for a given set of strain tensor E_{ij} we can construct a quadric surface with its centre at P_0 give by

$$E_{ij}\xi_i\xi_j = 1 \text{..........} (68)$$

This quadric surace is known as Strain Quadric.

Prop.-1. *The extension of a line element through the centre of a strain quadric in the direction of any central radius vector is equal to the inverse of the square of the radius vector.*

Let, $L = P_0Q_0$ and (N_1, N_2, N_3) be the direction cosines of P_0Q_0. Let (ξ_1, ξ_2, ξ_3) be the co-ordinates of Q_0 and $E_{(N)}$ be the extension of the line element P_0Q_0 in the direction of P_0Q_0. Then

$$E_{(N)} = E_{ij}N_jN_i \text{.............} (69)$$

Also, $$N_i = \frac{\xi_i}{L} \text{....................} (70)$$

$$\therefore E_{(N)} = \frac{E_{ij}\xi_i\xi_j}{L^2} = \frac{1}{L^2} \text{ (using (68))}$$

Prop.-2 . *The displacement of a material point at any point on the strain quadric relative to that at the centre is directed along the normal to the surface of the quadric at that point.*

Let $\bar{u}_i$ be the displacement of the material point at Q_0 relative to that at P_0 due to strain deformation only. Then

$$\bar{u}_i = E_{ij}\xi_{ij} \text{ (using (66))........} (71)$$

since ξ_i are relative co-ordinates of Q_0 w.r.t. P_0.

We consider the quadric function

$$2G(\xi_1,\xi_2,\xi_3) = E_{ij}\xi_i\xi_j \text{ .... } (72)$$

So, equation of strain quadric reduces to

$$2G(\xi_1,\xi_2,\xi_3) = 1 \text{ .... } (73)$$

Also from (71), $\dfrac{\partial}{\partial \xi_i}(2G) = \dfrac{\partial}{\partial \xi_i}(E_{kl}\xi_k\xi_l) = E_{kl}\left[\dfrac{\partial \xi_k}{\partial \xi_i}\xi_l + \dfrac{\partial \xi_l}{\partial \xi_i}\xi_k\right]$

$$= E_{kl}\left[\delta_{ki}\xi_l + \delta_{li}\xi_k\right] = \left[E_{kl}\delta_{ki}\xi_l + E_{kl}\delta_{li}\xi_k\right]$$

$$= E_{il}\xi_l + E_{ki}\xi_k = E_{ij}\xi_j + E_{ji}\xi_j$$

$$= E_{ij}\xi_j + E_{ij}\xi_j = 2E_{ij}\xi_j$$

$$\therefore \frac{\partial G}{\partial \xi_i} = E_{ij}\xi_j = \bar{u}_i \text{ .... } (74)$$

But $\dfrac{\partial G}{\partial \xi_i}$ are direction ratios of the normal to the quadric surface (73) at the point Q_0, which follows that relative displacement is directed along the normal to the quadric surface at Q_0.

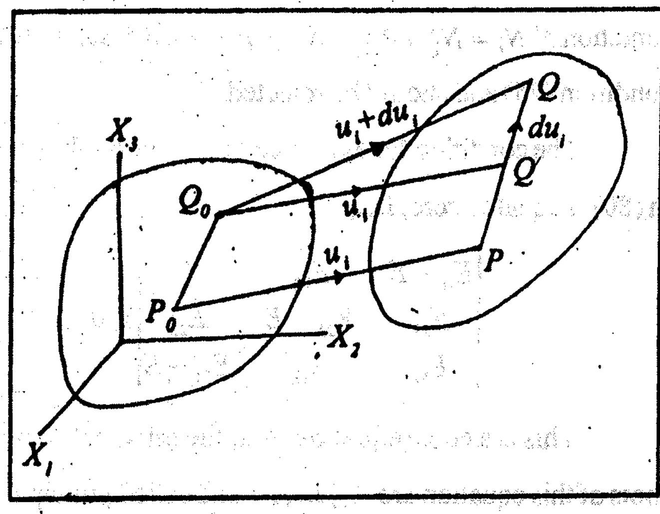
2.7 Principal Strain :

When the direction of a line element at a given point of a continuum remains unchanged by strain deformation then that direction is called principal direction of strain or principal axis of strain and the extension that occurs along the principal direction is called 'Principal Strain'.

If (X_1, X_2, X_3) and $(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3)$ be the co-ordinates of two neighbouring material points P_0 and Q_0 such that $P_0Q_0 = dL$ and oriented in the direction (N_1, N_2, N_3), then

$$\left.\begin{aligned} N_i &= \frac{dX_i}{dL}, i = 1,2,3 \\ \text{and} \quad N_iN_i &= 1 \end{aligned}\right\} \text{ .... } (75)$$

Let E_{ij} be the strain tensor at P_0 and in the deformed state P_0 and Q_0 moves to the points $P(x_1, x_2, x_3)$



and $Q(x_1+dx_1, x_2+dx_2, x_3+dx_3)$ respectively. Also let u_i and u_i+du_i are the displacements of P_0 and Q_0 respectivey.

If the line element P_0Q_0 is to be the principal direction of strain at P_0, then its direction will remain unchanged due to strain deformation, i.e., PQ must be parallel to P_0Q_0. Hence, we must have the relative displacement du_i is proportional to dX_i, i.e.,

$$du_i \propto dX_i, i = 1,2,3$$

or, $$du_i = EdX_i, i = 1,2,3 \quad \text{.......... (76)}$$

where E is the constant.

$\therefore E = \dfrac{du_i}{dX_i} = \dfrac{dx_i - dX_i}{dX_i}$ = extension of the component dX_i per unit length and is thus extension of the element P_0Q_0 in the direction of P_0Q_0. This E is called principal strain.

Now the strain vector,

$$E_i^{(N)} = \frac{du_i}{dL} = \frac{du_i}{dX_i}\cdot\frac{dX_i}{dL} = EN_i \text{ (using (75), (76))} \quad \text{.......... (77)}$$

Also, the strain vector is related to strain tensor, i.e.,

$$E_i^{(N)} = E_{ij}N_j$$

Hence, $$EN_i = E_{ij}N_j \quad \text{..................................(78)}$$

or, $$E\delta_{ij}N_j = E_{ij}N_j$$

or, $$(E_{ij} - E\delta_{ij})N_j = 0, \quad i = 1,2,3 \quad \text{......................(79)}$$

Expamding in details,

$$\left.\begin{aligned}(E_{11}-E)N_1 + E_{12}N_2 + E_{13}N_3 = 0\\ E_{21}N_1 + (E_{22}-E)N_2 + E_{23}N_3 = 0\\ E_{31}N_1 + E_{32}N_2 + (E_{33}-E)N_3 = 0\end{aligned}\right\} \quad \text{........ (80)}$$

which is a set of three homogeneous equation in three unknowns N_1, N_2, N_3 which together with the condition $N_i N_i = N_1^2 + N_2^2 + N_3^2 = 1$. The trival solution $N_1 = 0, N_2 = 0, N_3 = 0$ of (80) is not compatible with the condition (75) and should be rejected.

The condition for existance of non-zero solution of (80) is that the determinant of the co-efficients of N_j in (80) is equal to zero, i.e.,

$$\begin{vmatrix} E_{11}-E & E_{12} & E_{13} \\ E_{21} & E_{22}-E & E_{23} \\ E_{31} & E_{32} & E_{33}-E \end{vmatrix} = 0 \quad \text{.........(81)}$$

This is a cubic equation of E, the principal strain. This equation is known as characteristic equation. The roots of this equation are E_1, E_2, E_3 and called principal strain. With each of these roots one can solve (80) using (75) to find the corresponding principal axis of strain, i.e., direction cosines (N_1, N_2, N_3) for three principal strains.

Theorem: *All principal strains are real.*

Proof. To show the three roots E_1, E_2, E_3 of the charactaristin equation (81) are real, let us suppose that one root, say, E_1 is complex. Then of course the complex conjugate E_1^* of E_1 is also a root of equation (81), as E_{ij} is real. Then from (78), we have

$$E_{ij} N_j^{(1)} = E_1 N_i^{(1)}, \quad i=1,2,3 \text{.........(82)}$$

and $$E_{ij} N_j^{*(1)} = E_1^* N_i^{*(1)}, \quad i=1,2,3 \text{.........(83)}$$

where $N_i^{(1)}$, i= 1,2,3 represent the direction cosines of the corresponding principal axis and $N_i^{*(1)}$, i=1,2,3, its complex conjugate.

Multiplying both sides of (82) and (83) by $N_i^{*(1)}$ and $N_i^{(1)}$ respectively, then we get

$$\left.\begin{aligned} E_{ij} N_j^{(1)} N_i^{*(1)} &= E_1 N_i^{(1)} N_i^{*(1)} = E_1 \sum_i \left|N_i^{(1)}\right|^2 \\ \text{and} \quad E_{ij} N_j^{*(1)} N_i^{(1)} &= E_1^* N_i^{*(1)} N_i^{(1)} = E_1^* \sum_i \left|N_i^{(1)}\right|^2 \end{aligned}\right\} \text{.......: (84)}$$

Now, $E_{ij} N_j^{*(1)} N_i^{(1)} = E_{ji} N_i^{*(1)} N_j^{(1)}$ (interchanging dummy suffixes)

$$= E_{ji} N_i^{*(1)} N_j^{(1)} (\because E_{ij} \text{ is symetric})$$

Then, (84) follows that

$$E_1 N_i^{(1)} N_i^{*(1)} = E_1^* N_i^{(1)} N_i^{*(1)}$$

or, $$E_1 \sum_i \left|N_i^{(1)}\right|^2 = E_1^* \sum_i \left|N_i^{(1)}\right|^2$$

or, $$\left(E_1 - E_1^*\right) \sum_i \left|N_i^{(1)}\right|^2 = 0 \text{ .... } (85)$$

Since $N_i^{(1)} N_i^{*(1)} = \sum \left|N_i^{(1)}\right|^2$ = sum of squares of real numbers $\neq 0$ unless all $N_i^{(1)}, i = 1,2,3$ are zero, hence,

$$E_1 = E_1^*$$

which implies imaginary part of $E_1 = 0$, i.e. E_1 is real.

Theorem : ***Principal directions of strain corresponding to distinct principal strains are orthogonal to each other.***

Proof. To show that three principal directions are mutually perpendicular let E_1, E_2, E_3 be three distinct roots of the characteristic equation (81). Also these are principal strains and the direction cosines of the corresponding principal axes are given by $N_i^{(1)}, N_i^{(2)}, N_i^{(3)} (i = 1,2,3)$. Now we have, from (78),

$$\left.\begin{aligned} E_{ij} N_j^{(1)} &= E_1 N_i^{(1)}, \; i = 1,2,3 \\ E_{ij} N_j^{(2)} &= E_2 N_i^{(2)}, \; i = 1,2,3 \\ E_{ij} N_j^{(3)} &= E_3 N_i^{(3)}, \; i = 1,2,3 \end{aligned}\right\} \text{ .... } (86)$$

Multiplying first and second equation of (86) by $N_i^{(2)}$ and $N_i^{(1)}$ respectively, we get

$$E_{ij} N_j^{(1)} N_i^{(2)} = E_1 N_i^{(1)} N_i^{(2)}$$

and $$E_{ij} N_j^{(2)} N_i^{(1)} = E_2 N_i^{(2)} N_i^{(1)}$$

Since, $E_{ij} N_j^{(2)} N_i^{(1)} = E_{ji} N_i^{(2)} N_j^{(1)}$, interchanging dummy suffixes

$$= E_{ij} N_i^{(2)} N_j^{(1)}, \text{ since } E_{ij} = E_{ji}, \text{ symmetric}$$

$$\therefore E_1 N_i^{(1)} N_i^{(2)} = E_2 N_i^{(2)} N_i^{(1)}$$

or, $(E_1 - E_2) N_i^{(1)} N_i^{(2)} = 0$

Since the roots are distinct, i.e., $E_1 \neq E_2$, therefore we must have

$$N_i^{(1)} N_i^{(2)} = 0 \quad \text{............ (87)}$$

which shows that $N_i^{(1)}$ and $N_i^{(2)}$ are orthogonal for $E_1 \neq E_2$. Thus two principal directions of strain corresponding to two distinct principal strains are orthogonal. Similar results are obtained for other set of pair of roots. Hence the principal directions of strain corresponding to distinct principal strains are orthogonal to each other.

2.8 Strain Invariants :

The individual components of strain tensor E_{ij} have geometrical meaning dependent on the choice of co-ordinate system, there are a number of combinations of E_{ij} which remain unaltered by a rotation of the co-ordinate system. They are called strain invariants.

To find the invariants we consider the characteristic cubic equation (81), for principal strain E :

$$\begin{vmatrix} E_{11} - E & E_{12} & E_{13} \\ E_{21} & E_{22} - E & E_{23} \\ E_{31} & E_{32} & E_{33} - E \end{vmatrix} = 0$$

or, $E^3 - E^2\theta + E\theta_2 - \theta_3 = 0$, say,........ (88)

where the coefficients are

$$\theta = E_{11} + E_{22} + E_{33}, \quad \text{........... (89.a)}$$

$$\theta_2 = E_{11}E_{22} + E_{22}E_{33} + E_{33}E_{11} - E_{12}^2 - E_{23}^2 - E_{31}^2$$

$$= \begin{vmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{vmatrix} + \begin{vmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{vmatrix} + \begin{vmatrix} E_{33} & E_{31} \\ E_{13} & E_{11} \end{vmatrix}, \quad \text{................(89.b)}$$

$$\theta_3 = E_{23}^2 E_{11} + E_{31}^2 E_{22} + E_{12}^2 E_{33} - E_{11}E_{22}E_{33} - 2E_{12}E_{23}E_{31}$$

$$= \begin{vmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{vmatrix} \quad \text{........ (89.c)}$$

Since E_1, E_2, E_3 are the roots of the equation (88), they by relation between roots and co-efficients of the equation $\theta_1, \theta_2, \theta_3$ are sum of the product of roots taken one, two and three at a time given by

$$\left.\begin{aligned} \theta &= E_1 + E_2 + E_3 \\ \theta_2 &= E_1E_2 + E_2E_3 + E_3E_1 \\ \theta_3 &= E_1E_2E_3 \end{aligned}\right\} \quad \text{........ (90)}$$

Since the principal strains E_1, E_2, E_3 at a point have a geometrical meaning independent of the choice of co-ordinate system, it is clear that from (90) that $\theta_1, \theta_2, \theta_3$ given by (89.a), (89.b), (89.c) are invariant w.r.t. orthogonal transformation of co-ordinates and are respectively called first, second and third strain invariants.

Obsrvation : The first strain invariant $\theta = E_{11} + E_{22} + E_{33} = u_{i,i}$ represents the change in volume per unit original volume i.e., cubical dilatation or volumetric strain.

Isochoric Deformation : Isochoric deformation is one in which volume element remains unaltered. So, for isochoric deformation,

$$E_{11} + E_{22} + E_{33} = 0$$

Note-1. The extremum values of normal strain or extensional strain at a point of continuum are principal strains.

Note-2. The maximum shearing strain at any point of the continuum is equal to the one-half the difference between algebraically the largest and smallest principal strains at that point and that the corresponding direction bisects the angle between the largest and smallest principal directions.

2.9 Compatibility Equations for libear strain :

The strain components E_{ij} are given by

$$E_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right] = \frac{1}{2}\left[u_{i,j} + u_{j,i}\right]$$

where $u_{i,j} = \dfrac{\partial u_i}{\partial X_j}$.

It is clear that E_{ij} is functions of co-ordinates and three unknown $u_i, i = 1,2,3$ to be determined from

(91). Since E_{ij} is symmetric, so we have a system of six linear partial differential equations (91) to determine three unknows. Here we have three more equations than unknowns. For this reason, these equations may not, in general, have single-valued solutions u_i for an arbitrary choice of strain components and to ensure the existence of single-valued displacement solutions there must necessarily be subjected to additional restrictions or conditions, which implies that strain components must be compatible. These additional partial differential equations to be satisfied by strain components. E_{ij} alone which will ensure the existence of a set of single-valued displacements u_i are called "equations of compatibility"for strain components.

For this purpose, we try to eliminate u_i from (91).

Now,

$$E_{ij,kl}=\frac{\partial^2 E_{ij}}{\partial X_k \partial X_l}=\frac{1}{2}\left[u_{i,jkl}+u_{j,ikl}\right]\text{............ (92)}$$

Again,

$$E_{kl}=\frac{1}{2}\left[u_{k,l}+u_{l,k}\right]$$

$$\therefore E_{kl,ij}=\frac{1}{2}\left[u_{k,lji}+u_{l,kij}\right]\text{............ (93)}$$

From (92) and (93), we have

$$E_{ij,kl}+E_{kl,ij}=\frac{1}{2}\left[u_{i,jkl}+u_{j,ikl}+u_{k,lji}+u_{l,kji}\right]\text{......... (94)}$$

Interchanging, j and k in (94), we get

$$E_{ik,jl}+E_{jl,ik}=\frac{1}{2}\left[u_{i,kjl}+u_{k,ijl}+u_{j,lik}+u_{l,jik}\right]\text{......... (95)}$$

Now, since

$$u_{i,jkl}=u_{i,kjl};u_{l,kij}=u_{l,jik};u_{j,ikl}=u_{j,lik};u_{k,lij}=u_{k,ijl}$$

therefore from (94) and (95), by substraction, we get

$$E_{ij,kl}+E_{kl,ij}-E_{ik,jl}-E_{jl,ik}=0\text{ .... (96)}$$

This is compatibality equation for strain components and is necessary condition for the existence of single-valued displacement. These are Saint Venant's compatibility relations for strain component.

Equation (96) is a system of $3^4 = 81$ equations. Equations of which only six are algebraically independent (no one of them can be derived algebraically from other equation) because of symmetry in i, j and k, l. These six compatibility equations are

$$Q_{ij} = 0$$

where,
$$\left.\begin{aligned}
Q_{11} &= \left(E_{22,33} + E_{33,22}\right) - 2E_{23,23} \\
Q_{22} &= \left(E_{33,11} + E_{11,33}\right) - 2E_{31,31} \\
Q_{33} &= \left(E_{11,22} + E_{22,11}\right) - 2E_{12,12} \\
Q_{23} &= Q_{32} = E_{11,23} - \left(-E_{23,1} + E_{31,2} + E_{12,3}\right)_{,1} \\
Q_{31} &= Q_{13} = E_{22,31} - \left(E_{23,1} - E_{31,2} + E_{12,3}\right)_{,2} \\
Q_{12} &= Q_{21} = E_{33,12} - \left(E_{23,1} + E_{31,2} - E_{12,3}\right)_{,3}
\end{aligned}\right\} \quad \text{........ (97)}$$

When these conditions are satisfied by strain components E_{ij}, then single-valued displacement solutions of (91) exist.

Note 1. In two dimensions, there exists only one compatibility equation

$$Q_{33} = 0, \text{ i.e., } E_{11,22} + E_{22,11} = 2E_{12,12}$$

Note 2. Another set of equations which are equavalent to (97) but more useful, is given by setting $l = k$ in (96),

$$E_{ij,kk} + E_{kk,ij} - E_{ik,jk} - E_{jk,ik} = 0.$$

This set of equations contains only 9 equations of which only 6 are independent because of symmetry in i, j.

Note 3. Three completely independent compatibality equations for strain components are given by

$$Q_{23,23} = 0, Q_{31,31} = 0 \text{ and } Q_{12,12} = 0$$

or,
$$E_{11,2323} = \left(-E_{23,1} + E_{31,2} + E_{12,3}\right)_{,123}$$

$$E_{22,3131} = \left(E_{23,1} - E_{31,2} + E_{12,3}\right)_{,123}$$

$$E_{33,1212} = \left(E_{23,1} + E_{31,2} - E_{12,3}\right)_{,123}$$

These are fourth order.

2.10 Deformation Gradient Tensor :

In Lagrangian description method we have from (1),

$$x_i = x_i(X_1, X_2, X_3), i = 1,2,3$$

$$\therefore dx_1 = \frac{\partial x_1}{\partial X_1}dX_1 + \frac{\partial x_1}{\partial X_2}dX_2 + \frac{\partial x_1}{\partial X_3}dX_3 = \frac{\partial x_1}{\partial X_j}dX_j, j = 1,2,3$$

$$dx_2 = \frac{\partial x_2}{\partial X_1}dX_1 + \frac{\partial x_2}{\partial X_2}dX_2 + \frac{\partial x_2}{\partial X_3}dX_3 = \frac{\partial x_2}{\partial X_j}dX_j, j = 1,2,3$$

$$dx_3 = \frac{\partial x_3}{\partial X_1}dX_1 + \frac{\partial x_3}{\partial X_2}dX_2 + \frac{\partial x_3}{\partial X_3}dX_3 = \frac{\partial x_3}{\partial X_j}dX_j, j = 1,2,3$$

which gives $dx_i = \sum_{j=1}^{3}\frac{\partial x_i}{\partial X_j}dX_j, i = 1,2,3.$

$$\therefore [x_{i,j}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}$$

i.e., $[x_{i,j}]$ is a 3×3 non-zero matrix related to (X_1, X_2, X_3) system of co-ordinates,and by quotient law, it follows that $x_{i,j}$ are components of a non-zero tensor.This tensor is recognized as the gradient of a vector $\vec{x}$, the gradient being taken w.r.t. X_i called the deformatiom gradient tensor in the material form.

Note: Jacobian of the deformation is

$$|J| = \det[x_{i,j}] = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix}$$

2.11 Unit Summary :

In this unit we have discussed mainly the analysis part of the Strain for continuous media depending relative displacement, strain vector, rotation vector, strain anadric etc.

2.12 Worked Out Examples

Ex-1. *At a point the strain tensor is given by* $(E_{ij}) = \begin{pmatrix} 1 & 3 & -2 \\ 3 & 1 & -2 \\ -2 & -2 & 6 \end{pmatrix}$. *Determine the principal strains and principal directions of strain. Find also maximum value of normal strain and shearing strain.*

Ans. Principal strains of the given system is given by

$$\begin{vmatrix} 1-E & 3 & -2 \\ 3 & 1-E & -2 \\ -2 & -2 & 6-E \end{vmatrix} = 0$$

or, $(E-8)(E+2)(E-2)=0$

$\therefore E = 8, -2, 2$

Hence principal strains are 8, –2, 2.

Let $E_1 = 8, E_2 = -2, E_3 = 2$.

Now the principal directions are give by

$$(1-E)N_1 + 3N_2 - 2N_3 = 0$$

$$3N_1 + (1-E)N_2 - 2N_3 = 0$$

$$-2N_1 - 2N_2 + (6-E)N_3 = 0$$

Let $(N_1^{(1)}, N_2^{(1)}, N_3^{(1)})$ be the principal direction corresponding to principal strain $E_1 = 8$. Then we have

$$-7N_1^{(1)} + 3N_2^{(1)} - 2N_3^{(1)} = 0$$

$$3N_1^{(1)} - 7N_2^{(1)} - 2N_3^{(1)} = 0$$

$$-2N_1^{(1)} - 2N_2^{(1)} - 2N_3^{(1)} = 0$$

which gives, $N_1^{(1)} = -\frac{1}{\sqrt{6}}, N_2^{(1)} = -\frac{1}{\sqrt{6}}, N_3^{(1)} = -\frac{1}{\sqrt{6}}$.

Similarly for the other principal strains $E_2 = -2, E_3 = 2$ we get the principal directions, respectively,

$$N_1^{(2)} = \frac{1}{\sqrt{2}}, N_2^{(2)} = \frac{1}{\sqrt{2}}, N_3^{(2)} = 0;$$

$$N_1^{(3)} = \frac{1}{\sqrt{3}}, N_2^{(3)} = \frac{1}{\sqrt{3}}, N_3^{(3)} = -\frac{1}{\sqrt{3}}.$$

Maximum normal strain= maximum principal strain =8.

Maximum shearing strain $= \frac{1}{2}$ [maximum strain – minimum strain] $= \frac{1}{2}\left[8-(-2)\right] = 5.$

Ex.-2 *The strain tensor at a point is given by* $(E_{ij}) = \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Find the principal axes of strain and corresponding direction ratios of principal strains.*

Ans. The principal strains are the roots of the equation

$$\begin{vmatrix} a-E & b & 0 \\ b & -a-E & 0 \\ 0 & 0 & -E \end{vmatrix} = 0$$

or, $E\left[E^2 - (a^2 + b^2)\right] = 0$

or, $E = 0, \pm\sqrt{a^2 + b^2}.$

Hence the principal strains are $E_1 = 0, E_2 = \sqrt{a^2 + b^2}, E_3 = -\sqrt{a^2 + b^2}.$

The principal directions are given by

$$(a-E)N_1 + bN_2 = 0$$

$$bN_1 - (a+E)N_2 = 0$$

$$-EN_3 = 0.$$

Let $\left(N_1^{(1)}, N_2^{(1)}, N_3^{(1)}\right)$ be the direction ratios of the principal axis corresponding to $E_1 = 0$. Then we have

$$aN_1^{(1)} + bN_2^{(1)} = 0$$

$$bN_1^{(1)} - aN_2^{(1)} = 0$$

and $\quad 0 \cdot N_3^{(1)} = 0.$

which gives $N_1^{(1)} = 0, N_2^{(1)} = 0, N_3^{(1)} = 1.$

For $E_2 = \sqrt{a^2+b^2}$, let $\left(N_1^{(2)}, N_2^{(2)}, N_3^{(2)}\right)$ be the direction ratios of the principal axis.

$$\therefore \left(a - \sqrt{a^2+b^2}\right) N_1^{(2)} + bN_2^{(2)} = 0$$

$$bN_1^{(2)} - \left(a + \sqrt{a^2+b^2}\right) N_2^{(2)} = 0$$

$$\sqrt{a^2+b^2}\, N_3^{(2)} = 0.$$

Which gives $N_1^{(2)} = \dfrac{\left(a + \sqrt{a^2+b^2}\right)}{b}, N_2^{(2)} = 1, N_3^{(2)} = 0.$

Similarly for $E_3 = -\sqrt{a^2+b^2}$, we get

$$N_1^{(3)} = \frac{a - \sqrt{a^2+b^2}}{b}, N_2^{(3)} = 1, N_3^{(3)} = 0.$$

Ex.-3. *The displacement in an elastic solid is given by* $u_1 = a\left(X_1 + 2X_2 + 3X_3\right)$, $u_2 = a\left(-2X_1 + X_2\right)$, $u_3 = a\left(X_1 + 4X_2 + 2X_3\right)$, *where a is small quantity. Find dilatation, roation vector, shear, principal strain and corresponding principal axes.*

Ans. The strain tensor is

$$\left(E_{ij}\right) = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} \text{ where } E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right).$$

$$= \begin{pmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2} + \dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3} + \dfrac{\partial u_3}{\partial X_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_1} + \dfrac{\partial u_1}{\partial X_2}\right) & \dfrac{\partial u_2}{\partial X_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3} + \dfrac{\partial u_3}{\partial X_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial X_1} + \dfrac{\partial u_1}{\partial X_3}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial X_2} + \dfrac{\partial u_2}{\partial X_3}\right) & \dfrac{\partial u_3}{\partial X_3} \end{pmatrix} = \begin{pmatrix} a & 0 & 2a \\ 0 & a & 2a \\ 2a & 2a & 2a \end{pmatrix}.$$

Dilatation $= E_{11} + E_{22} + E_{33} = \frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_3} = a + a + 2a = 4a.$

$$\text{Rotation vector} = \vec{R} = \text{rot } \vec{u} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_3} \\ u_1 & u_2 & u_3 \end{pmatrix}$$

$$= \hat{i}\left(\frac{\partial u_3}{\partial X_2} - \frac{\partial u_2}{\partial X_3}\right) + \hat{j}\left(\frac{\partial u_1}{\partial X_3} - \frac{\partial u_3}{\partial X_1}\right) + \hat{k}\left(\frac{\partial u_2}{\partial X_1} - \frac{\partial u_1}{\partial X_2}\right)$$

$$= \hat{i}(4a - 0) + \hat{j}(3a - a) + \hat{k}(-2a - 2a)$$

$$= 4a\hat{i} + 2a\hat{j} - 4a\hat{k}.$$

Shear about X_1 axis by an angle $2E_{23} = 4a$,

about X_2 axis by an angle $2E_{31} = 4a$.

Principal strains are the roots of the equation

$$\begin{vmatrix} a-E & 0 & 2a \\ 0 & a-E & 2a \\ 2a & 2a & 2a-E \end{vmatrix} = 0$$

or, $\quad (E-a)\left(E^2 - 3aE - 6a^2\right) = 0$

$$\therefore E = a, \frac{a}{2}\left(3 \pm \sqrt{33}\right).$$

Direction ratios of the principal axes are given by

$$(a-E)N_1 + 2aN_3 = 0$$

$$(a-E)N_2 + 2aN_3 = 0$$

$$2aN_2 + 2aN_2 + (2a-E)N_3 = 0.$$

Hence the direction ratios of the principal axes corresponding to the principal strains $E_1 = a$, $E_2 = \frac{a}{2}\left(3+\sqrt{33}\right)$, $E_3 = \frac{a}{2}\left(3-\sqrt{33}\right)$ are $(1,-1,0)$, $\left(1,1,\frac{8}{\sqrt{33}-1}\right)$, $\left(1,1,-\frac{8}{\sqrt{33}+1}\right)$, respectively.

Ex.-4 *For the deformation defined by equations* $x_1 = X_1 + X_2$, $x_2 = X_1 - 2X_2$, $x_3 = X_1 + X_2 - X_3$. *Find the deformation gradients and Jacobian.*

Ans. Deformation gradient is given by

$$(x_{i,J}) = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

Also Jacobian $J = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = 3.$

To find $(X_{I,j})$, we solve the given set of equations. We get, by Crammer's rule,

$$X_1 = \frac{\begin{vmatrix} x_1 & 1 & 0 \\ x_2 & -2 & 0 \\ x_3 & 1 & -1 \end{vmatrix}}{J} = \frac{1}{3}(2x_1 + x_2),$$

$$X_2 = \begin{vmatrix} 1 & x_1 & 0 \\ 1 & x_2 & 0 \\ 1 & x_3 & -1 \end{vmatrix} \div J = \frac{1}{3}(x_1 - x_2),$$

$$X_3 = \begin{vmatrix} 1 & 1 & x_1 \\ 1 & -2 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} \div J = \frac{1}{3}(3x_1 - 3x_3) = x_1 - x_3.$$

$\therefore$ deformation gradient in the spatial form is

$$\left(x_{i,J}\right)=\begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{pmatrix}=\begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{3} & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Ex-5. *For the deformation defined by the equation* $X_1=\frac{1}{2}\left(x_1^2+x_2^2\right); X_2=\tan^{-1}\left(\frac{x_2}{x_1}\right), x_1\neq 0; x_3=x_3$. *Find* $\left(X_{i,J}\right),\left(x_{i,J}\right)$. *Show that the deformation is an isochoric deformation.*

Ans. The deformation gradient tensor, in the spatial form, is $\left(X_{i,J}\right)$ and given by

$$\left(X_{i,J}\right)=\begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{pmatrix}=\begin{pmatrix} x_1 & x_2 & 0 \\ \frac{x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Also from given set of equations we have

$$x_1=\sqrt{2X_1}\cos X_2, x_2=\sqrt{2X_1}\sin X_2, x_3=X_3.$$

$\therefore$ deformation gradient tensor in material form is

$$\left(x_{i,J}\right)=\begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix}=\begin{pmatrix} \frac{\cos X_2}{\sqrt{2X_1}} & -\sqrt{2X_1}\sin X_2 & 0 \\ \frac{\sin X_2}{\sqrt{2X_1}} & \sqrt{2X_1}\cos X_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}:$$

$$\text{Now, } J = |x_{i,j}| = \begin{vmatrix} \frac{\cos X_2}{\sqrt{2X_1}} & -\sqrt{2X_1}\sin X_2 & 0 \\ \frac{\sin X_2}{\sqrt{2X_1}} & \sqrt{2X_1}\cos X_2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos^2 X_2 + \sin^2 X_2 = 1.$$

Since, $J=1$, so the deformation is isochoric deformation.

Ex-6. *Show that the following are not possible strains components :*

$$E_{11} = e_{11} = k\left(x_1^2 + x_2^2\right), E_{22} = e_{22} = k\left(x_2^2 + x_3^2\right), E_{12} = e_{12} = k' x_1 x_2 x_3,$$

$$E_{13} = e_{13} = 0, E_{23} = e_{23} = 0, E_{33} = e_{33} = 0, \text{ where } k \text{ and } k' \text{ are constants.}$$

Ans. The given numbers $E_{ij} = e_{ij}$ are possible strain components only if each of six compatibility conditions are satisfied. Substituting the given e_{ij} into the condition

$$E_{11,22} + E_{22,11} = 2E_{22,12}$$

we have

$$2k = 2k' x_3$$

Since k and k' are constants, so the above condition can not be satisfied for $x_3 \neq 0$. For $x_3 = 0$, above equation gives $k=0$ and then all $E_{ij} = 0$. Hence, the given E_{ij} are not possible strain components.

Ex.-7. *If* $e_{11} = k\left(x_1^2 - x_2^2\right), e_{22} = k\left(x_2 x_2\right), e_{12} = k' x_1 x_2, e_{13} = e_{23} = e_{33} = 0$, *where* k, k' *are constants. Find the corresponding displacement, given that* $u_3 = 0$.

Ans. From given set of relations we have

$$e_{11,22} = -2k, e_{22,11} = 0, e_{12,12} = k'$$

and $e_{11,12} + e_{22,11} = 2e_{12,12}$ is satisfied iff

$$k = -k'$$

$\Rightarrow$ that e_{ij} are possible strain components only when $k = -k'$.

To find u_1, u_2 we have

$$e_{11} = u_{1,1} = k\left(x_1^2 - x_2^2\right) \text{.......(i)}$$

and $e_{22} = u_{2,2} = k\, x_1 x_2$ .... (ii)

$$e_{12} = \frac{1}{2}\left(u_{1,2} + u_{2,1}\right) = k' x_1 x_2 = -kx_1 x_2 \text{ .... (iii)}$$

$$e_{13} = \frac{1}{2}\left(u_{1,3} + u_{3,1}\right) = 0 \text{ .... (iv)}$$

$$e_{23} = \frac{1}{2}\left(u_{2,3} + u_{3,2}\right) = 0 \text{ .... (v)}$$

$$e_{33} = u_{3,3} = 0. \text{ .... (vi)}$$

But given that $u_3 = 0$. Therefore from (iv) and (v) we get

$$\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0 \text{ i.e., } \frac{\partial u_1}{\partial x_3} = 0$$

and $$\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 \text{ i.e., } \frac{\partial u_2}{\partial u_3} = 0$$

which implies that u_1 and u_3 are independent of x_1.

From (i), $$\frac{\partial u_1}{\partial x_1} = k\left(x_1^2 - x_2^2\right)$$

Integrating, $$u_1 = k\left[\frac{x_1^3}{3} - x_1 x_2^2\right] + f\left(x_2\right) \text{........ (vii)}$$

From (ii), $$\frac{\partial u_2}{\partial x_2} = kx_1 x_2$$

Integrating, $$u_2 = \frac{k}{2} x_1 x_2^2 + g\left(x_1\right) \text{........ (vii)}$$

Using (vii) and (viii) in (iii), we get

$$g'\left(x_1\right) = \frac{1}{2}kx_2^2 - f'\left(x_2\right)$$

This condition is possible only if

$$g'\left(x_1\right) = c, \frac{1}{2}kx_2^2 - f'\left(x_2\right) = c.$$

Then we get

$$g(x_1) = cx_1 + c_1$$

and $$f(x_2) = \frac{k}{6}x_2^3 - cx_2 + c_1$$

Hence, $$u_1 = \frac{1}{6}k\left(2x_1^3 - 6x_1x_2^2 + x_2^3\right) - cx_2 + c_1$$

$$u_2 = \frac{1}{3}kx_1x_2^2 + cx_1 + c_2.$$

Which are the displacement components associated with the given e_{ij} when the compatibility conditions are obeyed.

Ex.-8.

i) *Find the compatibility condition for the strain tensor (e_{ij}) if e_{11}, e_{22}, e_{12} are indepedent of x_3 and $e_{13} = e_{23} = e_{33} = 0$.*

ii) *Find the strain tensor, rotation tensor for the small deformation $\vec{u} = \alpha x_1x_2(\hat{e}_1 + \hat{e}_2) + 2\alpha(x_1 + x_2)x_3\hat{e}_3$ where α is a constant.*

Ans. i) Since e_{11}, e_{22}, e_{12} are indepdent of x_3 and $e_{13} = e_{23} = e_{33} = 0$, we can find on verification of six compatibility conditions that the only one condition to be satisfied is

$$e_{11,12} + e_{22,11} = 2e_{12,12}$$

ii) From given displacement vector we have its components

$$u_1 = \alpha x_1x_2,\ u_2 = \alpha x_1x_2 \text{ and } u_3 = 2\alpha(x_1 + x_2)x_3.$$

$$\therefore (e_{ij}) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \alpha x_2 & \frac{1}{2}(\alpha x_1 + \alpha x_2) & \alpha x_3 \\ \frac{1}{2}(\alpha x_1 + \alpha x_2) & \alpha x_1 & \alpha x_3 \\ \alpha x_3 & \alpha x_3 & 2\alpha(x_1 + x_2) \end{pmatrix}.$$

and $(R_{ij}) = \begin{pmatrix} 0 & \frac{1}{2}(\alpha x_1 - \alpha x_2) & -\alpha x_3 \\ -\frac{1}{2}(\alpha x_1 - \alpha x_2) & 0 & -\alpha x_3 \\ \alpha x_3 & \alpha x_3 & 0 \end{pmatrix}$ where $R_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$.

$$= \alpha \begin{pmatrix} 0 & \frac{1}{2}(x_1 - x_2) & -x_3 \\ \frac{1}{2}(x_1 - x_2) & 0 & -x_3 \\ x_3 & x_3 & 0 \end{pmatrix}.$$

Ex.-9. **Given $E_{11} = 5 + X_1^2 + X_2^2 + X_1^4 + X_2^4$, $E_{22} = 6 + 3(X_1^2 + X_2^2) + X_1^4 + X_2^4$, $E_3 = 0$, $E_{12} = 5 + 2X_1X_2(X_1^2 + X_2^2 + 2)$, $E_{23} = 0, E_{31} = 0$; determine it the system of strain is possible. If this strain distribution is possible, determine the displacement components in terms of assuming that the displacement and rotation at the origin are zero.**

For $E_{33} = 0$, so we have $\frac{\partial u_3}{\partial X_3} = 0 \Rightarrow u_3 = F_3(X_1, X_2)$ .... (i)

$$E_{31} = 0 \Rightarrow \frac{1}{2}\left(\frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_3}\right) = 0 \Rightarrow \frac{\partial u_1}{\partial X_3} = -\frac{\partial u_3}{\partial X_1} = -\frac{\partial F_3}{\partial X_1} \text{ (using (i))}$$

$$\therefore u_1 = -X_3 \frac{\partial F_3}{\partial X_1} + F_1(X_1, X_2) \text{ .... (ii)}$$

$$E_{23} = 0 \Rightarrow \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) = 0 \Rightarrow \frac{\partial u_2}{\partial X_3} = -\frac{\partial u_3}{\partial X_2} = -\frac{\partial F_3}{\partial X_2} \text{ (using (i))}$$

$$\therefore u_2 = -X_3 \frac{\partial F_3}{\partial X_2} + F_2(X_1, X_2) \text{ .... (iii)}$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) = 5 + 2X_1X_2(X_1^2 + X_2^2 + 2)$$

$$\Rightarrow \frac{1}{2}\left(-X_3\frac{\partial^2 F_3}{\partial X_2 \partial X_1}+\frac{\partial F_1}{\partial X_2}-X_3\frac{\partial^2 F_3}{\partial X_1 \partial X_2}+\frac{\partial F_2}{\partial X_1}\right)=5+2X_1X_2\left(X_1^2+X_2^2+2\right)$$

or, $$-2X_3\frac{\partial^2 F_3}{\partial X_1 \partial X_2}+\left(\frac{\partial F_1}{\partial X_2}+\frac{\partial F_2}{\partial X_1}\right)=10+4X_1X_2\left(X_1^2+X_2^2+2\right)$$

Equating both sides with coefficient of X_3, then we get

$$\frac{\partial^2 F_3}{\partial X_1 \partial X_2}=0 \text{.......... (iv)}$$

and $$\frac{\partial F_1}{\partial X_2}+\frac{\partial F_2}{\partial X_1}=10+4X_1X_2\left(X_1^2+X_2^2+2\right) \text{........... (v)}$$

$$E_{11}=\frac{\partial u_1}{\partial X_1}=5+X_1^2+X_2^2+X_1^4+X_2^4$$

or, $$-X_3\frac{\partial^2 F_3}{\partial X_1^2}+\frac{\partial F_1}{\partial X_1}=5+\left(X_1^2+X_2^2\right)+\left(X_1^4+X_2^4\right) \text{ (using (ii))}$$

Equating both sides with coefficient of X_3, then we get

$$\frac{\partial^2 F_3}{\partial X_1^2}=0 \text{.......... (vi)}$$

and $$\frac{\partial F_1}{\partial X_1}=5+\left(X_1^2+X_2^2\right)+\left(X_1^4+X_2^4\right) \text{.......... (vii)}$$

$$E_{22}=\frac{\partial u_2}{\partial X_2}=6+3\left(X_1^2+X_2^2\right)+\left(X_1^4+X_2^4\right)$$

or, $$-X_3\frac{\partial^2 F_3}{\partial X_2^2}+\frac{\partial F_2}{\partial X_2}=6+3\left(X_1^2+X_2^2\right)+\left(X_1^4+X_2^4\right)$$

Equating both sides with coefficient of X_3, then we get

$$\frac{\partial^2 F_3}{\partial X_2^2}=0 \text{.......... (viii)}$$

and $$\frac{\partial F_2}{\partial X_2}=6+3\left(X_1^2+X_2^2\right)+\left(X_1^4+X_2^4\right) \text{........ (ix)}$$

Now, from (iv), (vii) and (viii) it is clear that F_3 is a linear function of X_1 and X_2 i.e.

$$F_3 = a_3 + b_3 X_2 + c_3 X_1 \text{ .... (x)}$$

From (vii), by integrating,

$$F_1 = 5X_1 + \frac{X_1^3}{3} + X_1 X_2^2 + \frac{X_1^5}{5} + X_1 X_2^4 + f_1(X_2) \text{ ...... (xi)}$$

Similarly from (ix),

$$F_2 = 6X_2 + 3X_1^2 X_2 + X_2^3 + X_1^4 X_2 + \frac{X_2^5}{5} + f_2(X_1) \text{ ...... (xii)}$$

Using (xi) and (xii) into (v), we get

$$2X_1X_2 + 4X_1X_2^3 + \frac{\partial f_1}{\partial X_2} + 6X_1X_2 + 4X_1^3X_2 + \frac{\partial f_2}{\partial X_1} = 10 + 4X_1^3 + 4X_1X_2^3 + 8X_1X_2 + 4X_1^3X_2$$

$$\Rightarrow \frac{\partial f_1}{\partial X_2} + \frac{\partial f_2}{\partial X_1} = 10 \text{ ....(xiii)}$$

$$\Rightarrow \frac{\partial f_2}{\partial X_1} = 10 - \frac{\partial f_1}{\partial X_2}$$

which implies that each expression is a constant, and

let $\quad \frac{\partial f_1}{\partial X_2} = C_1$ ....(xiv)

then $\quad \frac{\partial f_2}{\partial X_1} = 10 - C_1$ ....(xv)

From (xiv), (xv) on integrating, we get

$$\left.\begin{aligned} & f_1 = a_1 + C_1 X_2 \\ \text{and} \quad & f_2 = a_2 + (10 - C_1) X_1 \end{aligned}\right\} \text{ ....(xvi)}$$

Using (xvi) in (xi) and (xii), we get

$$\left.\begin{aligned} & F_1 = 5X_1 + \frac{X_1^3}{3} + X_1X_2^2 + \frac{X_1^5}{5} + X_1X_2^4 + a_1 + C_1X_2 \\ & F_2 = 6X_2 + 3X_1^2X_2 + X_2^3 + X_1^4X_2 + \frac{X_2^5}{5} + (10 - C_1)X_1 + a_2 \\ \text{and} \quad & F_3 = a_3 + b_3X_2 + C_3X_1 \end{aligned}\right\} \text{ ....(xvii)}$$

Using (xviii), into (i),(ii) and (iii), we get

$$\left.\begin{aligned} u_1 &= -C_3X_3 + 5X_1 + \frac{X_1^3}{3} + X_1X_2^2 + \frac{X_1^5}{5} + X_1X_2^4 + C_1X_2 + a_1, \\ u_2 &= -b_3X_3 + 6X_2 + 3X_1^2X_2 + X_2^3 + X_1^4X_2 + \frac{X_2^5}{5} + (10 - C_1)X_1 + a_2, \\ u_3 &= C_3X_1 + b_3X_2 + a_3 \end{aligned}\right\} \text{..............(xviii)}$$

Given that displacement is zero at origin. So, initially

$$X_1 = 0, X_2 = 0, X_3 = 0, u_1 = 0, u_2 = 0, u_3 = 0.$$

Which implies from (xviii),

$$a_1 = 0, a_2 = 0, a_3 = 0 \text{.....................(xix)}$$

Also, initially rot $\vec{u} = \vec{0}$ at origin, i.e.,

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_3} \\ u_1 & u_2 & u_3 \end{vmatrix} = \vec{0} \quad \text{when } X_1 = 0, X_2 = 0, X_3 = 0.$$

i.e., $2b_3\hat{i} - 2C_3\hat{j} + \left[(10 - C_1) + 6X_1X_2 + 4X_1^3X_2 - 2X_1X_2 - 4X_1X_2^3 - C_1\right]\hat{k} = \vec{0}$

when $X_1 = 0, X_2 = 0, X_3 = 0,$

i.e., when $X_1 = 0, X_2 = 0, X_3 = 0,$ then $b_3 = 0, C_3 = 0, 10 - 2C_1 = 0$ i.e., $C_1 = 5.$

Hence, $u_1 = 5X_1 - \frac{X_1^3}{3} + X_1X_2^2 + \frac{X_1^5}{5} + X_1X_2^4 + 5X_2,$

$$u_2 = 6X_1 + 3X_1^2X_2 + X_2^3 + X_1^4X_2 + \frac{X_2^5}{5} + 5X_1,$$

$$u_3 = 0.$$

2.13 Self Assessment Questions :

1. Given the strain field $(E_{ij}) = \begin{pmatrix} k_1X_2 & 0 & 0 \\ 0 & -k_2X_2 & 0 \\ 0 & 0 & -k_2X_2 \end{pmatrix}$. What should be the relation between k_1 and k_2

such that there will be no volume change.

2. Calculate the strain invariants from strain tensor $\left(E_{ij}\right)=\begin{pmatrix}5 & -1 & -1\\ -1 & 4 & 0\\ -1 & 0 & 4\end{pmatrix}$. Determine the principal strains.

Obtain strain invariants from them. Show the equivalence of strain invariants.

3. Determine the principal directions and principal strains for $\left(E_{ij}\right)=\begin{pmatrix}e & e & e\\ e & e & e\\ e & e & e\end{pmatrix}$.

4. For deformation $u_1 = AX_1 + BX_1\left(X_1^2+X_2^2\right)^{-1}$, $u_2 = AX_2 + BX_2\left(X_1^2+X_2^2\right)^{-1}$, $u_3 = CX_3$ where A, B, C are constants. Find $\left(E_{ij}\right), \left(R_{ij}\right)$. Also find the principal values and principal axes of E_{ij}.

5. The displacement in an elastic solid is given by $u_1 = X_1 + 2X_2 + 3X_3$, $u_2 = -2X_1 + 5X_2$, $u_3 = X_1 + 4X_2 - 3X_3$. Find dilatation, rotation vector, shear, principal strain and corresponding principal axes.

6. The strain components are given by $E_{11} = E_{22} = E_{33} = \alpha F\left(X_1, X_2, X_3\right)$, $E_{12} = E_{23} = E_{31} = 0$, where α is a constant. Show that in order to be compatible, F must be a linear function of $\left(X_1, X_2, X_3\right)$.

7. Under what conditions the following is a possible system of strain

$$E_{11} = a + b\left(X_1^2+X_2^2\right) + X_1^4 + X_2^4, E_{22} = \alpha + \beta\left(X_1^2+X_2^2\right) + X_1^4 + X_2^4,$$

$$E_{12} = A + BX_1X_2\left(X_1^2+X_2^2+C\right), E_{31} = 0 = E_{32} = E_{33}.$$

8. Show that $E_{11} = k\left(X_1^2+X_2^2\right), E_{22} = kX_2^2, E_{12} = kX_1X_2, E_{33} = E_{23} = E_{31} = 0$ is a possible state of strain while $E_{11} = kX_3\left(X_1^2+X_2^2\right), E_{22} = kX_2^2X_3, E_{12} = kX_1X_2X_3, E_{33} = E_{23} = E_{31} = 0$ is not a possible one.

9. For the small deformation defined by the equations

$x_1 = X_1 + \alpha X_3, x_2 = X_2, x_3 = X_3 - \alpha X_1$, where α is a small non zero constant. Find dilatation. Find the condition under which the deformation is isochoric.

10. Find the constants α and β such that the small deformation defined by $u_1 = \alpha x_1 + 3x_2$, $u_2 = x_1 - \beta x_2$, $u_3 = 3x_3$ is isochoric.

11. Find the nature of the function f such that $e_{11} = \alpha f(x_2, x_3)$, $e_{22} = e_{33} = \beta f(x_2, x_3), e_{12} = e_{23} = e_{33} = 0$, where α and β are constants, is a possible system of strain.

12. If e_{13} and e_{23} are the only non-zero strain components and e_{13} and e_{23} are independent of x_3; show that the compatibility conditions may be reduced to the following single conditions:

$$e_{13,2} - e_{23,1} = \text{Constant}$$

13. If $e_{11} = e_{22} = e_{33} = e_{12} = 0, e_{13} = \frac{\partial \varphi}{\partial x_2}$ and $e_{23} = \frac{\partial \varphi}{\partial x_1}$ where φ is a function of x_1 and x_2; show that φ must satisfy the equation $\nabla^2 \varphi = \text{constant}$.

14. For a certain small deformation, the displacement gradient at a point is given by $(u_{i,j}) = \begin{pmatrix} 4 & 1 & 4 \\ -1 & -4 & 0 \\ 0 & 2 & 6 \end{pmatrix}$.

Find the strain components, the strain invariants, the principal strains and principal direction of strain.

2.14 Further Suggested Readings :

1. Continuum Mechanics : T.J. Chung, Prentice-Hall.
2. Schaum's Outline of theory and problem of Continuum Mechanics: Gedrge R. Mase, McGraw-Hill.
3. Continuum Mechanics: A.J.M. Spencer, Longman.
4. Mathematical Theory of Continuum Mechanics: R.N. Chatterjee, Narosa Publishing House.
5. Foundation of Fluid Mechanics : S.W. Yuan, Prentice Hall.
6. Fluid Dynamics: J.K. Goyal, K.P. Gupta, Pragati Prakashan.
7. Textbook of Fluid Dynamics: F. Chorlton, CBS Publishers and Distributors.
8. Theory of Elasticity : Yu. Amenzade, Mir Publishers, Moscow.
9. Applied Elasticity : C.T. Wang, McGraw-Hill.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I
Paper-V : Group – A : Marks – 50

Module No. - 51
Mechanics of Continuous Media

STRUCTURE :

3.1 Introduction:

The deformation which have discussed earlier is generally caused by external forces that give rise to interactions between neighbouring portions in the interior parts of a continuum. Such interactions are studied through the concept of stress. This portion deals with the theory of stress.

3.2 Objectives:

Most important part in continuum mechanics is stress. Here we explain how the stress acts on the material. The students will learn the technique and method for stress.

3.3 Key Words

Stress, body force, surface force, Cauchy's stress principle, stress vector, stress tensor, equation of continuity, equation of motion, stress quadric, stress transformation, principal stress, principal directions, stress invariants.

3.4 Body Forces and Surface Forces:

In continuum mechanics, two distinct types of forces are considered: (i) *internal forces*, and (ii) *external forces*.

Internal are bounding forces of interaction between constituent particles belonging to the body to ensure the existence of the strength while external are forces exerted by any agent external to the body. Internal forces act even when no external forces are applied to the body.

The external forces that cause the continuum body to be deformed are : (i) *volume or body forces and* (ii) *surface forces*.

Surface forces are short-range forces and arise from the action of one body on another through a contact surface when two bodies are in direct contact. Surface forces also arise from the action of one part of the body on an adjacent part across a common boundary surface. It act only on the surface element and is thus proportional to the area of the surface element. Surface forces are specified as force per unit area.

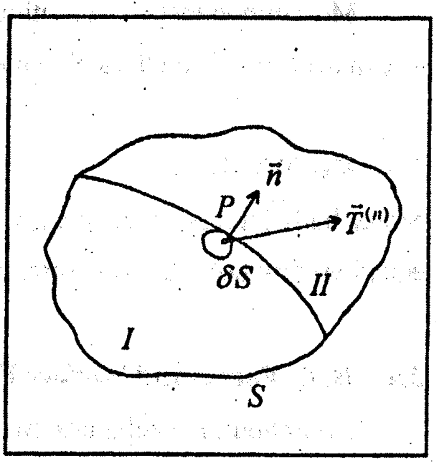
Body (volume or mass) forces are long-range forces that arise from the action of one body on another when they are at a distance from each other. They are capable of penetrating into the deep interior of a body and act equally on all the matter within an element of volume and are proportional to the mass contained in the volume element. They are specified as force per unit mass.

Examples: (i) Gravitational force and magnetic forces are examples of body forces, (ii) hydrostatic pressure of liquid, pressures of one solid body on another due to contact.

3.5 Stress Vector:

The concept of stress arises from the consideration of the internal forces which the particles of one part of the deformed body exerts on the particles of the adjacent part through the separating boundary surface in the form of restoring forces.

Let us consider a deformed body every part of which is held in equilibrium under the action of external forces. Consider the body of surface area S which is divided into two parts I and II within the body. Now part I will be in equilibrium under the action of remaining external forces acting on this part of the body and the internal bounding forces of interaction transmitted by the particles of the material in part II outside S on the particles of the material in part I inside S across the surface S. These internal forces acting at the points of S are now converted into external surface forces relative to part I. Let $P(x_1, x_2, x_3)$ be any point on S and δS be an element of arbitrary size and shape surrounding point P with outward unit normal $\vec{n}$ at P directed from part I to part II. Now the surface forces distributed over the surface element δS of S can be reduced into a single restoring force $\delta\vec{F}^{(n)}$ acting at P along a definite direction together with a couple $\delta\vec{G}^{(n)}$. As $\delta S \to 0$ then the ratio $\dfrac{\delta\vec{F}^{(n)}}{\delta S} \to a$ definite limit $\vec{T}^{(n)}(x_1, x_2, x_3)$, a force per unit area at P, called *stress vector* while $\dfrac{\delta\vec{G}^{(n)}}{\delta S} \to a$ limit $\vec{M}^{(n)}$, called *couple stress vector*, i.e.,



$$\underset{\delta S \to 0}{Lt} \frac{\delta\vec{F}^{(n)}}{\delta S} = \vec{T}^{(n)},$$

and $$\underset{\delta S \to 0}{Lt} \frac{\delta\vec{G}^{(n)}}{\delta S} = \vec{M}^{(n)}.$$

For most materials, this couple stress vector, $\vec{M}^{(n)} = \vec{0}$ and these type of continuum bodies are called *nonpolar*.

Now we shall confine ourselves to this type of material. Obviously the sterss vector $\vec{T}^{(n)}$ depends on the positional co-ordinates (x_1, x_2, x_3) and on the orientation of the particular surface element δS through P whose exterior unit normal is $\vec{n}$. For some differently oriented surface element passing through the same point P, having different unit normal, the stress vector at P will, in general, be different. Thus stress vector $\vec{T}^{(n)}(x_1,x_2,x_3)$ is the external surface force per unit area acting at a point (x_1,x_2,x_3) on the surface element with normal $\vec{n}$ representing the action of material outside the surface element on the material inside the surface element.

It is important to remember that in general $\vec{T}^{(n)}$ is not directed along the dierction of normal $\vec{n}$. Stress vector $\vec{T}^{(n)}$ can be resolved into two components:

i) the normal component directed along the normal $\vec{n}$, called 'normal stress' and denoted by $N^{(n)}$, and

ii) the tangential component directed along the tangent to the surface element, called 'tangential stress' and denoted by $S^{(n)}$.

The tangential stress is sometimes called ***shearing stress***. The normal stress $N^{(n)}$ is positive if its sense coincides with the sense of outward normal to the surface element at a given point.

If $\vec{T}^{(n)}$ represents the action of the part II on part I transmitted through a surface element with normal $\vec{n}$ at the given point (x_1,x_2,x_3) and the stress vector $\vec{T}^{(-n)}$ represents the reaction of part I on part II transmitted through the same surface element at (x_1,x_2,x_3), then by Newton's law of action and reaction

$$\vec{T}^{(-n)}(x_1,x_2,x_3) = -\vec{T}^{(n)}(x_1,x_2,x_3)$$

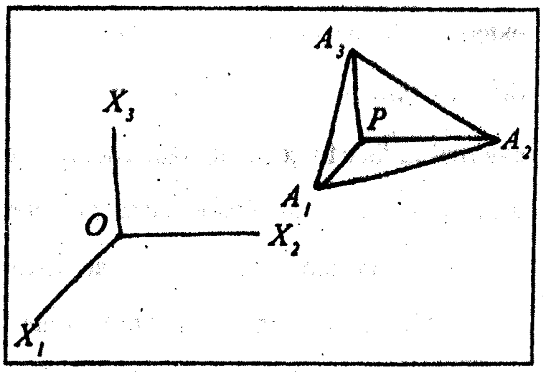
i.e., the stress vector acting on the opposite sides of the same surface element at a given point are equal in magnitude but opposite in sign.

3.6 Components of Stress Vector:

Since infinite number of planes can be drawn through P, we get an infinite number of stress vector acting at P. Therefore, to completely specify the stress at a point, we have to know all the stress vectors across all the plane elements through the point. But it is not necessary to specify each stress vector corresponding to every plane, it is sufficient to know the three stress vectors, acting on any three mutually perpendicular planes through the given point of the medium.

Theorem. *The stress vector at a point on any arbitrary plane surface is a linear function of three stress vectors acting on any three mutually perpendicular planes through that point.*

Proof. Let $P(x_1,x_2,x_3)$ be a point of the deformed continuum body. Let us consider a small tetrahedron $PA_1A_2A_3$, imagined to be isolated from the medium, having one vertex at P with three orthogonal faces



$PA_2A_3, PA_1A_3, PA_1A_2$ parallel to the co-ordinate planes and an arbitrary oblique plane $A_1A_2A_3$ with unit normal $\vec{n}$ at a small distance h from P. Consider the motion of the specific portion of the continuum which occupies the tetrahedron at time t. Let ρ be density of this portion and $\delta S_1, \delta S_2, \delta S_3, \delta S$ the areas of the faces PA_2A_3, PA_1A_3, PA_1A_2, $A_1A_2A_3$ respectively. If (n_1, n_2, n_3) be the components of $\vec{n}$ and δV the volume of the tetrahedron, then we have

δS_1 = area of PA_2A_3 parallel to x_2x_3 plane

= projection of δS on the plane parallel to x_2x_3 plane

$= n_1\delta S$

Similarly, $\delta S_2 = n_2\delta S, \delta S_3 = n_3\delta S$

$$\therefore \delta S_i = n_i\delta S \text{ and } \delta V = \frac{1}{3}\cdot h\cdot\delta S \quad \text{............. (1)}$$

Now, the motion of the tetrahedron is governed by the body forces and the stress vector across the four boundary planes due to the material outside of it. Let $\vec{T}^{(n)}, \vec{T}_1, \vec{T}_2, \vec{T}_3$ be the average stress vector across the faces $A_1A_2A_3$, $PA_2A_3, PA_1A_3, PA_1A_2$ respectively and the stress vectors acting across the plane faces PA_2A_3, PA_1A_3, PA_1A_2 by the material outside of the tetrahedron will be $-\vec{T}_1, -\vec{T}_2, -\vec{T}_3$ respectively because the exterior normals to these planes are directed oppositely to the positive dierctions of the axes. If $\vec{F}$ denote the body force per unit mass acting on inside and $\vec{f}$ the acceleration of the material within tetrahedron per unit mass, then Cauchy's law of motion for the tetrahedron of the continuum is

$$\vec{T}^{(n)}\delta S - \vec{T}_1\delta S_1 - \vec{T}_2\delta S_2 - \vec{T}_3\delta S_3 + \vec{F}\rho\delta V = \rho\delta V\vec{f}$$

or, $$\vec{T}^{(n)}\delta S - \vec{T}_1\delta S_1 - \vec{T}_2\delta S_2 - \vec{T}_3\delta S_3 + \vec{F}\rho\frac{1}{3}h\delta S = \rho\vec{f}\frac{1}{3}h\delta S$$

or, $$\vec{T}^{(n)}\delta S - \vec{T}_1 n_1\delta S - \vec{T}_2 n_2\delta S - \vec{T}_3 n_3\delta S + \frac{1}{3}h\rho\left(\vec{F} - \vec{f}\right)\delta S = 0 \quad \text{[using (1)]}$$

Dividing both sides by δS and passing to the limit as $h \to 0$ while keeping the direction $\vec{n} = (n_1, n_2, n_3)$ constant, then we have

$$\vec{T}^{(n)} - \vec{T}_1 n_1 - \vec{T}_2 n_2 - \vec{T}_3 n_3 = \vec{0}$$

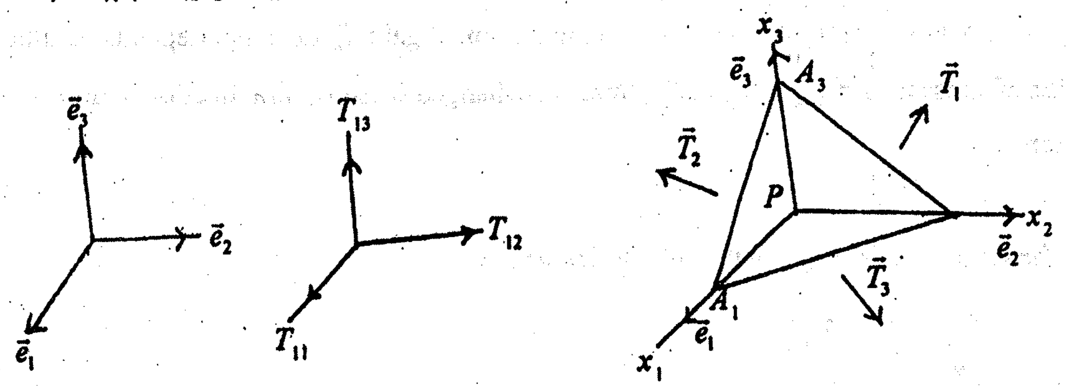
or, $$\vec{T}^{(n)} = \vec{T}_1 n_1 + \vec{T}_2 n_2 + \vec{T}_3 n_3 \quad \text{.................. (2)}$$

Thus, the stress vector at a point across the plane with normal $\vec{n}$ is a linear combinatino of the stress vectors acting on the three orthogonal planes through that point.

3.7 Stress Tensor:

Let us consider the plane PA_2A_3 normal to x_1 axis of the small tetrahedron $PA_1A_2A_3$. This plane cuts the medium into two portions: one side of the plane towards which positive direction of x_1 axis points is called positive side; other side is called negative side of the plane. Let $\vec{T}_1$ be the stress vector at P exerted across the plane PA_2A_3 by the material on the positive side of this plane on the material on the negative side of the plane. Now $\vec{T}_1$ can be resolved into two components: normal stress and shearing stress. The normal stress component of $\vec{T}_1$ which is directed along the positive dierction of x_1 axis is denoted by T_{11}. Similarly the shearing stress which acts in the plane PA_2A_3 can further be resolved into two components: along the positive of x_2 and x_3 axes and are denoted by T_{12}, T_{13} respectively. Thus, the stress vector $\vec{T}_1$ has three stress components T_{11}, T_{12}, T_{13} parallel to the co-ordinate axes. If $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be the unit vectors along the co-ordinate axes, then

$$\vec{T}_1 = T_{11}\vec{e}_1 + T_{12}\vec{e}_2 + T_{13}\vec{e}_3 = T_{1i}\vec{e}_i \quad \text{......... (3.1)}$$



Similarly, the stress vectors $\vec{T}_2, \vec{T}_3$ acting at P across the planes PA_3A_1, PA_1A_2 respectively can be resolved and if T_{21}, T_{22}, T_{23} and T_{31}, T_{32}, T_{33} be the components of $\vec{T}_2$ and $\vec{T}_3$ then we have

$$\vec{T}_2 = T_{21}\vec{e}_1 + T_{22}\vec{e}_2 + T_{23}\vec{e}_3 = T_{2i}\vec{e}_i \quad \text{................ (3.2)}$$

$$\vec{T}_3 = T_{31}\vec{e}_1 + T_{32}\vec{e}_2 + T_{33}\vec{e}_3 = T_{3i}\vec{e}_i \text{ .... (3.3)}$$

Combining (3.1), (3.2), (3.3), we get

$$\vec{T}_i = T_{ij}\vec{e}_j$$

Thus, T_{ij} is the jth component of the stress vector $\vec{T}_i$ at P acting across a plane normal to x_i axis. If $\vec{T}^{(n)}$ be the stress vector at P across a plane normal to $\vec{n}(n_1, n_2, n_3)$, then

$$\vec{T}^{(n)} = \vec{T}_i n_i = T_{ij}\vec{e}_j n_i = \vec{e}_j\left(T_{ij}n_i\right)$$

$$= \vec{e}_i\left(T_{ji}n_j\right) \text{ (interchanging dummy suffixes)}$$

or, $T_i^{(n)}\vec{e}_i = \vec{e}_i\left(T_{ji}n_j\right)$ where $T_i^{(n)}$ are components of $T^{(n)}$

$$\therefore T_i^{(n)} = T_{ji}n_j \text{ .... (4)}$$

Thus stress vector at P across any arbitrary plane normal to $\vec{n}(n_1, n_2, n_3)$ is a linear combination of nine stress components T_{ji} acting across three mutually perpendicular planes at the point parallel to the co-ordinate plane, i.e., the stress at any point of the medium is completely characterised by specification of nine scalar quantities T_{ji}.

Since n_j is an arbitrary vector and the product $T_{ji}n_j$ is also a vector being equal to $T_i^{(n)}$ then from the quotient law of tensor, it follows that nine quantities T_{ji} form a second order tensor called stress tensor at P. Hence stress vector is a linear function of stress tensor at that point. Again T_{ij} does not depends on time t and the transformation of stress tensor $T'_{pq} = \alpha_{ip}\alpha_{jq}T_{ij}$ remains unchanged when α_{ij} is a function of time. Hence stress tensor is objective.

Note 1. The stress matrix, (T_{ij}), at the point is denoted by

$$\left(T_{ij}\right) = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

and dimension of each stress component is *force/(length)²*.

Note 2. The stress components T_{11}, T_{22}, T_{33} are called *normal stresses*, and the remaining components $T_{12}, T_{13}, T_{21}, T_{23}, T_{31}, T_{32}$ are called *shearing stresses*.

Note 3. The normal stress on any arbitrary plane normal to $\vec{n}(n_1, n_2, n_3)$ is given by

$$N^{(n)} = T_i^{(n)} n_i = T_{ji} n_j n_i = T_{ij} n_i n_j \quad \text{.......................... (5)}$$

If $S^{(n)}$ be the magnitude of the shearing stress, then

$$T_1^{(n)^2} + T_2^{(n)^2} + T_3^{(n)^2} = N^{(n)^2} + S^{(n)^2}$$

$$\Rightarrow S^{(n)^2} = T_1^{(n)^2} + T_2^{(n)^2} + T_3^{(n)^2} - N^{(n)^2}$$

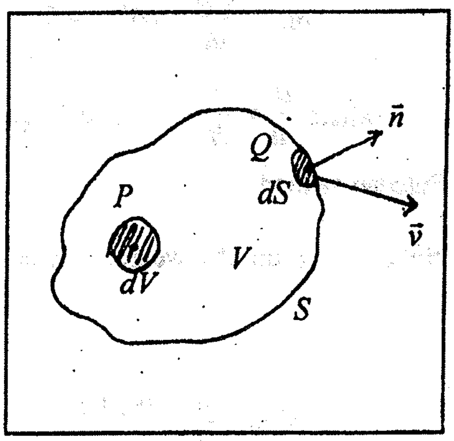
$$= (T_{j1} n_j)^2 + (T_{j2} n_j)^2 + (T_{j3} n_j)^2 - (T_{ij} n_i n_j)^2 \quad \text{...................... (6)}$$

Note 4. The stress tensor T_{ij} is called Cauchy's stress tensor. The relation (4) connecting the stress vector $\vec{T}^{(n)}$ and the stress tensor T_{ij} is known as Chauchy's law (hypothesis).

3.8 Equation of Continuity in Eulerian Method:

Equation of continuity in Eulerian method the principle of conservation of mass is expressed in the form that the rate at which the mass of the continuum within any fixed closed surface increases is equal to the rate at which the net mass of the continuum flows in across the boundary surface.

Letus considerany fixed closed arbitrary surface S enclosing a volume V lying entirely in a region through which continuum moves. Let $P(x_1, x_2, x_3)$ be any point in it and ρ be the density of the material at P at time t, so that, $\rho = \rho(x_1, x_2, x_3, t)$



Let dV be the element of volume at P and mass of this element is ρdV. Then total mass of the continuum which fills the volume V is $\iiint_V \rho dV$. Now, net local rate of the increase of this mass in V is $\frac{\partial}{\partial t}\iiint_V \rho dV = \iiint_V \frac{\partial \rho}{\partial t} \rho dV$, since V is a fixed region of space, and the space co-ordinates x_1, x_2, x_3 are independent of t.

Let Q be any point on the surface S, and dS be the element at Q with outward drawn unit normal vector $\vec{n}$. Let $\vec{v}$ be the velocity of the particle at Q at time t. Then $\vec{n}\cdot\vec{v}$ is the normal component of velocity $\vec{v}$ along the direction of outward normal and $\rho\vec{n}\cdot\vec{v}dS$ is the mass of the continuum leaving volume V by flowing out across dS per unit time. Hence, the mass of the continuum entering into V through dS per unit time is $(-\rho\vec{n}\cdot\vec{v}dS)$.

Then the rate of net mass of the continuum flows in across total boundary surface S is

$$\iint_S (-\rho\vec{n}\cdot\vec{v}dS) = -\iint_S \rho\vec{n}\cdot\vec{v}dS$$

Now from the principle of conservation of mass, we get

$$\iiint_V \frac{\partial\rho}{\partial t}dV = -\iint_S \rho\vec{n}\cdot\vec{v}dS$$

$$= -\iint_S \vec{n}\cdot(\rho\vec{v})dS$$

$$= -\iiint_V \vec{\nabla}\cdot(\rho\vec{v})dV \text{ (by Gauss's div. theorem)}$$

or, $$\iiint_V \left[\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v})\right]dV = 0$$

since V is an arbitrary volume, so, the integrand must vanish at every point of continuum, and hence

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \quad \text{.................... (7)}$$

or, $$\frac{\partial\rho}{\partial t} + (\vec{v}\cdot\vec{\nabla})\rho + \rho\vec{\nabla}\cdot\vec{v} = 0$$

or, $$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\rho + \rho\vec{\nabla}\cdot\vec{v} = 0$$

or, $$\frac{d\rho}{dt} + \rho\vec{\nabla}\cdot\vec{v} = 0$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}$, and equation (7) or equation (8) is the required equation of continuity in spatial i.e., Eulerian method.

Note 1. From (8), we get $\vec{\nabla}\cdot\vec{v} = \frac{1}{\rho}\frac{d\rho}{dt}$ = rate at which density is decreasing

= divergence of $\vec{v}$.

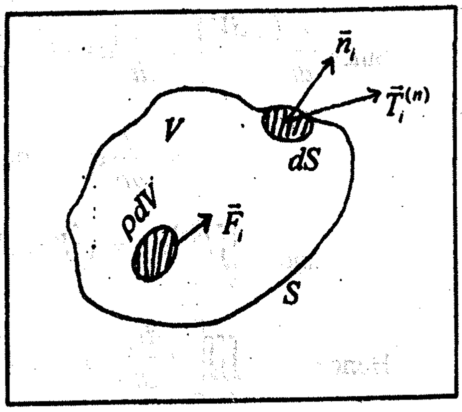
$$\vec{\nabla}\cdot\vec{v} = \frac{d(\log_e \rho)}{dt}$$

Note 2. Equation of continuity in material i.e., Lagrangian method is

$$\rho_0 = \rho \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)},$$

where $\rho_0 = \rho(X_1, X_2, X_3), \rho = \rho(X_1, X_2, X_3, t), x_i = x_i(x_1, x_2, x_3, t)$.

3.9 Principle of balance of linear momentum: Cauchy's first equations of motion.



The principle of balance of linear momentum can be expressed as : the time rate of change of total linear momentum of a specific portion of the continuum is equal to the resultant external force acting on the considered portion.

Let us consider a specific portion of a deformed continuum occupy an arbitrary volume V at time t bounded by the closed surface S. Let $\rho(x_1, x_2, x_3, t)$ and $v_i(x_1, x_2, x_3)$ be the density field and velocity field of the continuum. Then the total linear momentum $P_i(t)$ of the specific portion within the region V at time t is

$$P_i(t) = \iiint_V \rho v_i dV$$

Now, the external forces that acts on the considered portion consists of body force and surface force. Let $F_i(x_1, x_2, x_3, t)$ be the field of body force per unit mass and $T_i^{(n)}(x_1, x_2, x_3, t)$ be the field of surface force per unit area of the surface in the form of stress vector acting across the element of surface with outward unit normal vector n_i exerted by the surrounding material outside the volume V. Then the resultant external force $R_i(t)$ acting on the portion of continuum in V at time t is given by

$$R_i(t) = \iiint_V \rho F_i dV + \iint_S T_i^{(n)} dS$$

But $\frac{d P_i(t)}{dt} = R_i(t)$ for the balancing of linear momentum, for any choice of region V. Hence,

$$\frac{d}{dt}\left(\iiint_V \rho v_i dV\right) = \iiint_V \rho F_i dV + \iint_S T_i^{(n)} dS$$

or, $\iiint_V \rho \frac{dv_i}{dt} dV + \iiint_V v_i \frac{d(\rho dV)}{dt} = \iiint_V \rho F_i dV + \iint_S T_{ji} n_j dS$ [using (4)]

or, $\iiint_V \rho \frac{dv_i}{dt} dV = \iiint_V \rho F_i dV + \iiint_V T_{ji,j} dV$

Since $\frac{d(\rho dV)}{dt} = \frac{d\rho}{dt} dV + \rho \frac{d(dV)}{dt} = \frac{d\rho}{dt} dV + \rho \frac{d(JdV_0)}{dt} (\because dV = JdV_0)$

$$= \frac{d\rho}{dt} dV + \rho dV_0 \cdot \frac{dJ}{dt} = \frac{d\rho}{dt} dV + \rho dV_0 J \vec{\nabla} \cdot \vec{v} \left(\because \frac{dJ}{dt} = J \vec{\nabla} \cdot \vec{v} \right)$$

and $\iint_S T_i^{(n)} dS = \iint_S T_{ji} n_j dS = \iiint_V T_{ji,j} dV$ by Gauss's div. theorem.

Hence, $\iiint_V \left[\rho \frac{dv_i}{dt} - \rho F_i - T_{ji,j} \right] dV = 0$

Since V is an arbitrary volume, so

$$\rho \frac{dv_i}{dt} - \rho F_i - T_{ji,j} = 0 \quad \text{.................... (10.a)}$$

at every point of continuum. This equation of motion is called *Cauchy's First Equation* of motion.

Note 1. If $u_i(x_1, x_2, x_3, t)$ be the displacement field, then $v_i = \dot{u}_i = \frac{du_i}{dt}$, and equation of motion is $\rho \ddot{u}_i - \rho F_i - T_{ji,j} = 0$ .... (10.b)

Note 2. If the continuum is in static equilibrium, then the acceleration $\dot{v}_i = \frac{dv_i}{dt} = 0$ and hence the equation of equillibrium is

$$\rho F_i + T_{ji,j} = 0 \quad \text{...................... (11)}$$

Note 3. Equations of equllibrium in components form are:

$$\frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial y} + \frac{\partial T_{13}}{\partial z} + \rho X = 0,$$

$$\frac{\partial T_{21}}{\partial x} + \frac{\partial T_{22}}{\partial y} + \frac{\partial T_{23}}{\partial z} + \rho Y = 0,$$

$$\frac{\partial T_{31}}{\partial x} + \frac{\partial T_{32}}{\partial y} + \frac{\partial T_{33}}{\partial z} + \rho Z = 0,$$

and equations of motion along the co-ordinate axes are

$$\rho\frac{d^2u}{dt^2} = \rho X + \frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial y} + \frac{\partial T_{13}}{\partial z},$$

$$\rho\frac{d^2v}{dt^2} = \rho Y + \frac{\partial T_{21}}{\partial x} + \frac{\partial T_{22}}{\partial y} + \frac{\partial T_{23}}{\partial z},$$

$$\rho\frac{d^2w}{dt^2} = \rho Z + \frac{\partial T_{31}}{\partial x} + \frac{\partial T_{32}}{\partial y} + \frac{\partial T_{33}}{\partial z},$$

where $F_i = (X, Y, Z)$, and displacements $u_i = (u, v, w)$.

3.10 Principle of balance of angular momentum: Cauchy's second equation of motion; Symmetry of Stress tensor.

The principle of balance of angular momentum is *the time rate of change of total angular momentum of a specific portion of the continuum about an arbitrary point is equal to the resultant moment about the same point of external force acting on the consider portion of the continuum.*

The total angular momentum $\vec{H}(t)$ about the origin of the portion of the continuum in V at time t is given by

$$\vec{H}(t) = \iiint_V (\vec{r} \times \vec{v})\,\rho dV$$

or in component form

$$H_i(t) = \iiint_V e_{ijk} x_j \rho v_k dV \quad \text{..................... (12)}$$

$$\text{where, } e_{ijk} = \begin{cases} 0, \text{if any two of } i, j, k \text{ are equal} \\ 1, \text{ if } i, j, k \text{ are even permutation of } 1,2,3 \\ 1, \text{ if } i, j, k \text{ are odd permutation of } 1,2,3 \end{cases}$$

Now the resultant moment $\vec{M}(t)$ of external forces acting on the continuum at time t about origin is given by

$$\vec{M}(t) = \iiint_V (\vec{r} \times \vec{F})\,\rho dV + \iint_S \vec{r} \times \vec{T}^{(n)} dS$$

or in component form $M_i(t) = \iiint_V e_{ijk} x_j \rho F_k dV + \iint_S e_{ijk} x_j T_k^{(n)} dS$ .... (13)

From the principle of balance of angular momentum, we have

$$\frac{dH_i}{dt} = M_i$$

i.e. $\frac{d}{dt}\left(\iiint_V e_{ijk} x_j \rho v_k dV\right) = \iiint_V e_{ijk} x_j \rho F_k dV + \iint_S e_{ijk} x_j T_k^{(n)} dS$ .... (14)

Also, $\frac{d}{dt}\left(\iiint_V e_{ijk} x_j \rho v_k dV\right) = \iiint_V e_{ijk} \frac{d}{dt}\left(x_j \rho v_k dV\right)$, as e_{ijk} is constant

$$= \iiint_V e_{ijk} x_j v_k \frac{d(\rho dV)}{dt} + \iiint_V e_{ijk} \frac{d(x_j v_k)}{dt} \rho dV$$

$$= \iiint_V e_{ijk}\left(\dot{x}_j v_k + x_j \dot{v}_k\right) \rho dV \left(\because \frac{d(\rho dV)}{dt} = 0, \text{ using (9)}\right)$$

If $u_i(x_1, x_2, x_3, t)$ be the displacement field, and then

$$x_i = X_i + u_i$$

and $\dot{x}_i = \dot{u}_i = v_i$

$\therefore \frac{d}{dt}\left(\iiint_V e_{ijk} x_j \rho v_k dV\right) = \iiint_V e_{ijk} v_j v_k \rho dV + \iiint_V e_{ijk} x_j \dot{v}_k \rho dV$ .... (15)

Now, $e_{ijk} v_j v_k = e_{ikj} v_k v_j$, interchanging dummy suffixes

$$= - e_{ijk} v_k v_j, \because e_{ijk} = -e_{ikj}$$

$\therefore 2 e_{ijk} v_j v_k = 0$

and hence $\iiint_V e_{ijk} v_j v_k \rho dV = 0$ .... (16)

Again, $\iint_S e_{ijk} x_j T_k^{(n)} dS = \iint_S e_{ijk} x_j T_{lk} n_l dS \left(\because T_k^{(n)} = T_{lk} n_l\right)$

$$= \iiint_V div.\left(e_{ijk} x_j T_{lk}\right) dV \text{ (using Gauss's div. theorem)}$$

$$= \iiint_V e_{ijk}\left(x_j T_{lk}\right)_{,l} dV$$

$$= \iiint_V e_{ijk}\left(x_{j,l} T_{lk} + x_j T_{lk,l}\right) dV$$

$= \iiint_V e_{ijk}\left(\delta_{jl} T_{lk} + x_j T_{lk,l}\right) dV = \iiint_V e_{ijk}\left(T_{jk} + x_j T_{lk,l}\right) dV$ .... (17)

Using (15), (16), (17) in (14),

$$= \iiint_V e_{ijk} x_j \dot{v}_k \rho dV = \iiint_V e_{ijk} x_j \rho F_k dV + \iiint_V e_{ijk} \left(T_{jk} + x_j T_{lk,l}\right) dV$$

or, $$\iiint_V e_{ijk} T_{jk} dV = \iiint_V e_{ijk} x_j \left(\dot{v}_k \rho - \rho F_k - T_{lk,l}\right) dV$$

$$= \iiint_V e_{ijk} x_j \times 0 \times dV \quad \text{[using (10.a)]}$$

$$= 0$$

Since the volume V is an arbitrary, so at every point of the continuum we get,

$$e_{ijk} T_{jk} = 0 \quad \text{.....................} (18)$$

Put, $i=1$, then $e_{1jk} T_{jk} = 0$

which $\Rightarrow e_{123} T_{23} + e_{132} T_{32} = 0$

or, $T_{23} - T_{32} = 0 \left(\because e_{123} = -e_{132}\right)$

$\therefore T_{23} = T_{32}$

Similarly, when $i=2$, then we get $T_{31} = T_{13}$ and for $i=3$, $T_{12} = T_{21}$.

Therefore, we have $T_{ij} = T_{ji}$ .... (19)

which implies that the stress tensor is symmetric, and the stress at any point of the medium is completely specified by six rather than nine components of stress tensor.

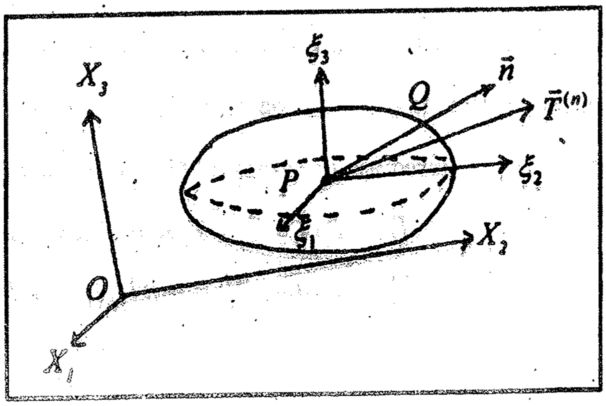
Using (19) in (10.b), then the Cauchy's First equation of motion takes the form

$$\rho \ddot{u}_i = \rho \dot{v}_i = \rho F_i + T_{ij,j} \quad \text{.....................} (20)$$

Equations (19) and (20) together called Cauchy's second equation of motion.

3.11 Stress Quadric of Cauchy:

The nature of distribution of stresses in the deformed body can be understand more clearly by a geometrical treatment.



Let OX_1, OX_2, OX_3 be a set of rectangular axes through O. Let T_{ij} be the stress tensor at $P(x_1, x_2, x_3)$. Now, we introduce a local system of axes $P\xi_1, P\xi_2, P\xi_3$ with origin at P and axes are parallel to OX_1, OX_2, OX_3

respectively. For a given set of stress tensor T_{ij} we can construct a quadric surface with its centre at *P* given by

$$T_{ij}\xi_i\xi_j = 1 \quad \text{.........................} (21)$$

This surface is called the *stress quadric of Cauchy.*

Property 1. The normal stress across any plane through the centre of stress quadric is equal to the inverse of the square of the central radius vector of the quadric normal to the plane.

Proof. Let T_{ij} be the stress tensor at $P(x_1, x_2, x_3)$ referred to a set of axes OX_1, OX_2, OX_3 fixed in space. Let $Q(\xi_1, \xi_2, \xi_3)$ be a point on the stress quadric $T_{ij}\xi_i\xi_j = 1$.

Next we consider a plane element through *P* normal to *PQ*. Let $|\overline{PQ}| = r$ and n_1, n_2, n_3 being the direction cosines of *PQ* where $\vec{n} = (n_1, n_2, n_3)$. Let $T_i^{(n)}$ denotes the stress vector across this plane, then

$$T_i^{(n)} = T_{ij}n_j$$

If $N^{(n)}$ be the normal stress at *P* across the plane normal to n_i, then

$$N^{(n)} = T_i^{(n)}n_i = T_{ij}n_in_j$$

But at *Q*, we have

$$\xi_i = rn_i \text{ i.e., } n_i = \frac{\xi_i}{r}$$

$$\therefore N^{(n)} = \frac{T_{ij}\xi_i\xi_{ij}}{r^2} = \frac{1}{r^2}$$

Hence the result.

3.12 Stress Transformation:

Let T_{ij} be the stress tensor at a point w.r.t. a system of axes OX_1, OX_2, OX_3. Let this system of axes be rotated about *O* to obtain a new system of axes OX_1', OX_2', OX_3'. Let T_{pq}' be the stress tensor w.r.t. the new system of axes. If (l_1, l_2, l_3) be direction consines of OX_1', (m_1, m_2, m_3) be the direction consines of OX_2' and (n_1, n_2, n_3) be direction consines of OX_3' then we have

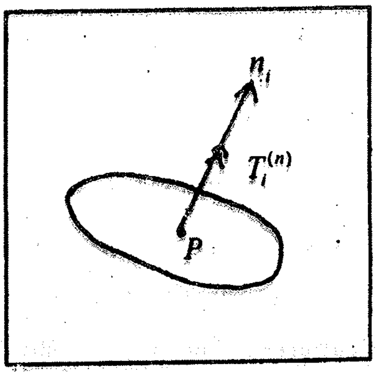
$$T_{11}' = T_{ij}l_il_j, T_{22}' = T_{ij}m_im_j, T_{33}' = T_{ij}n_in_j$$

$$T_{23}' = T_{ij}m_in_j, T_{31}' = T_{ij}n_il_j, T_{12}' = T_{ij}l_im_j$$

i.e., $\begin{pmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{pmatrix} = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix}$

i.e., $(T'_{ij}) = (\alpha)(T_{ij})(\alpha)^T$, $i, j = 1,2,3$ .... (22)

where T'_{ij} is the matrix consisting of the nine components of stress in the primed axis, (T_{ij}) is the matrix of stress components corresponding unprimed axis and (α) is the matrix formed by the direction consines of OX'_1, OX'_2, OX'_3 as elements $(\alpha)^T$ is its transpose.



3.13 Principal Stress:

From previous discussion it is clear that the stress vector on a surface element need not be collinear with the normal to the surface element. So, when the stress vector acts entirely in a direction perpendicular to the element of plane on which it acts, then it is called Principal Stress, and the element of plane area on which principal stress is acting is called principal plane, the direction of principal stress is called Principal direction of stress or principal axis of stress.

Let $P(x_1, x_2, x_3)$ be any point inside the continuum and $\vec{n} = (n_1, n_2, n_3)$ be the unit normal vector at $P, T_i^{(n)}$ the stress vector across the element about *P* of area whose magnitude is *T*. If $T_i^{(n)}$ be the principal stress, then it must be along the normal n_i, so,

$$T_i^{(n)} = Tn_i$$

with $n_i n_i = 1$

Also, from the relation between stress vector and stress tensor, we have

$$T_i^{(n)} = T_{ij} n_j$$

Hence, $T_{ij} n_j = Tn_i = Tn_j \delta_{ij}$

or, $(T_{ij} - T\delta_{ij}) n_j = 0$ .... (23)

Expanding in detail, we get

$$\left.\begin{aligned} (T_{11} - T)n_1 + T_{12}n_2 + T_{13}n_3 = 0 \\ T_{12}n_1 + (T_{22} - T)n_2 + T_{23}n_3 = 0 \\ \text{and } T_{31}n_1 + T_{32}n_2 + (T_{33} - T)n_3 = 0 \end{aligned}\right\} \quad \text{......... (24)}$$

with $n_i n_i = 1$.

Equations in (24) is a set of three homogeneous equations of n_1, n_2, n_3. When $n_1 = 0, n_2 = 0, n_3 = 0$ then we get trivial solution, which is not compatible with $n_i n_i = 1$. Hence for non-trivial solution of (24) we must have the determinant of the coefficients of this should be zero,

i.e. $$\begin{vmatrix} T_{11}-T & T_{12} & T_{13} \\ T_{21} & T_{22}-T & T_{23} \\ T_{31} & T_{32} & T_{33}-T \end{vmatrix} = 0 \quad \text{..................} (25)$$

which gives the cubic equation of T and known as *characteristic equation* for the determination of principal stresses. The three roots T_1, T_2, T_3 of (25) are *principal stresses*. Substituting the values of each of three principal stresses T_1, T_2, T_3 into (24) and using the relation $n_i n_i = 1$, one can find the three sets of solutions of n_1, n_2, n_3, i.e., direction cosines for three principal stresses.

Property 1. The principal stresses T_1, T_2, T_3 are real.

Property 2. The directions corresponding to the principal stresses T_1, T_2, T_3 are mutually perpendicular.

Proofs of the above results are same as that in the case of principal strains.

Note 1. If two roots of (25) are equal, the directions corresponding to these roots lie in a plane perpendicular to the direction corresponding to the simple root.

Note 2. If all three roots are equal then every direction of space may be taken as principal direction of stress.

3.14 Stress Invariants:

Principal stresses T_1, T_2, T_3 are the roots of (25) i.e..

$$\begin{vmatrix} T_{11}-T & T_{12} & T_{13} \\ T_{21} & T_{22}-T & T_{23} \\ T_{31} & T_{32} & T_{33}-T \end{vmatrix} = 0$$

or, $T^3 - \theta T^2 + \theta_2 T - \theta_3 = 0$

where
$$\left.\begin{aligned} \theta &= T_{11} + T_{22} + T_{33} \\ \theta_2 &= \begin{vmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{23} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{33} & T_{31} \\ T_{31} & T_{11} \end{vmatrix} \\ \theta_3 &= \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} \end{aligned}\right\} \quad \text{..................} (26)$$

Also from relations between the roots and coefficients are

$$\theta = T_1 + T_2 + T_3$$

$$\theta_2 = T_1T_2 + T_2T_3 + T_3T_1$$

$$\theta_3 = T_1T_2T_3$$

Since the principal stresses T_1, T_2, T_3 at a point do not depend on the choice of co-ordinate axes, therefore,

$$T_{11} + T_{22} + T_{33} = T_1 + T_2 + T_3 \text{ (First Invariant)},$$

$$\begin{vmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{23} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{33} & T_{31} \\ T_{31} & T_{11} \end{vmatrix} = T_1T_2 + T_2T_3 + T_3T_1 \text{ (Second Invariant)},$$

$$\begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = T_1T_2T_3 \text{ (Third Invariant)}$$

that implies that $\theta, \theta_2, \theta_3$ given by (26) are invariant w.r.t. an orthogonal transformation of co-ordinates.

3.15 Unit Summary:

In this module we have discussed about stress in simple way and methods, techniques are straight-forward which is suitable for the students.

3.14 Worked Out Examples:

Ex.1. *The stress tensor at P* is given by $(T_{ij}) = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$. *Determine principal stresses and principal directions.*

Ans. Let T be the principal stress value, then the characteristic equation for determining the principal stress value is

$$\begin{vmatrix} 3-T & 1 & 1 \\ 1 & -T & 2 \\ 1 & 2 & -T \end{vmatrix} = 0$$

or, $(3-T)(T^2-4) - 1(-T-2) + 1(2+T) = 0$

or, $T^3 - 3T^2 - 6T + 8 = 0$

or, $(T-1)(T^2 - 2T - 8) = 0$

or, $(T-1)(T-4)(T+2) = 0$

$\therefore T = 1, 4, -2$.

So, $T_1 = 1, T_2 = 4, T_3 = -2$ are the values of principal stress. Let (n_1, n_2, n_3) be the principal direction corresponding to the principal stress $T_1 = 1$ and which are given by the set of following equations

$$(3-1)\ n_1 + n_2 + n_3 = 0$$
$$n_1 - n_2 + 2n_3 = 0$$
$$n_1 + 2n_2 - n_3 = 0$$

with $n_1^2 + n_2^2 + n_3^2 = 1$

$\therefore n_1 = -n_3$ and $n_1 = -n_2$.

But $n_1^2 + n_2^2 + n_3^2 = 1$ gives $3n_1^2 = 1$ i.e., $n_3 = -\frac{1}{\sqrt{3}}$.

$$\therefore n_1 = \frac{1}{\sqrt{3}}, n_2 = -\frac{1}{\sqrt{3}}, n_3 = -\frac{1}{\sqrt{3}}.$$

Similarly let (l_1, l_2, l_3) and (m_1, m_2, m_3) be the principal directions corresponding to the principal stresses $T_2 = 4$ and $T_3 = -2$ respectively. Then we have the two sets of equations

$$(3-4)\ l_1 + l_2 + l_3 = 0$$
$$l_1 - 4l_2 + 2l_3 = 0$$
$$l_1 + 2l_2 = 4l_3 = 0 \text{ with } l_1^2 + l_2^2 + l_3^2 = 1$$

and
$$(3+2)\ m_1 + m_2 + m_3 = 0$$
$$m_1 + 2m_2 + 2m_3 = 0$$
$$m_1 + 2m_2 + 2m_3 = 0 \text{ with } m_1^2 + m_2^2 + m_3^2 = 1.$$

Solving above equations for non-trivial solutions we get

$$l_1 = \frac{2}{\sqrt{6}}, l_2 = \frac{1}{\sqrt{6}}, l_3 = \frac{1}{\sqrt{6}};\ m_1 = 0, m_2 = \frac{1}{\sqrt{2}}, m_3 = \frac{-1}{\sqrt{2}}.$$

Ex.2. *The stress matrix at a point P in a material is given by* $(T_{ij}) = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 4 & -5 & 0 \end{pmatrix}$. *Find (i) the stress vector on a plane element through P and parallel to the plane* $2x_1 + x_2 - x_3 = 1$, *(ii) the magnitude of the stress vector and (iii) the angle that the stress vector makes with the normal to the plane.*

Ans. Since the plane element on which the stress vector is requierd is parallel to the plane $2x_1 + x_2 - x_3 = 1$, so, direction cosines are $\left(\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$ and the components n_i of the unit normal to the plane element are

$n_1=\frac{2}{\sqrt{6}},n_2=\frac{1}{\sqrt{6}},n_3=-\frac{1}{\sqrt{6}}$.

$$\therefore T_1=T_{11}n_1+T_{12}n_2+T_{13}n_3=3\cdot\frac{2}{\sqrt{6}}+1\cdot\frac{1}{\sqrt{6}}+4\cdot\left(-\frac{1}{\sqrt{6}}\right)=\frac{3}{\sqrt{6}}$$

$$T_2=T_{21}n_1+T_{22}n_2+T_{23}n_3=1\cdot\frac{2}{\sqrt{6}}+2\cdot\frac{1}{\sqrt{6}}+(-5)\left(-\frac{1}{\sqrt{6}}\right)=\frac{9}{\sqrt{6}}$$

and $$T_3=T_{31}n_1+T_{32}n_2+T_{33}n_3=4\cdot\frac{2}{\sqrt{6}}+(-5)\cdot\frac{1}{\sqrt{6}}+0\cdot\left(-\frac{1}{\sqrt{6}}\right)=\frac{3}{\sqrt{6}}.$$

Hence $\vec{T}^{(n)}=T_1\vec{e}_1+T_2\vec{e}_2+T_3\vec{e}_3=\frac{3}{\sqrt{6}}(\vec{e}_1+3\vec{e}_2+\vec{e}_3)$

which is the required stress vector.

Magnitude of teh stress vector is $\left|\vec{T}^{(n)}\right|=\sqrt{\left(\frac{3}{\sqrt{6}}\right)^2+\left(\frac{9}{\sqrt{6}}\right)^2+\left(\frac{3}{\sqrt{6}}\right)^2}=\sqrt{\frac{99}{6}}=\sqrt{\frac{133}{2}}$.

Let θ be the angle between the stress vector $\vec{T}^{(n)}$ and the normal vector $\vec{n}$. Therefore,

$$\vec{T}^{(n)}\cdot\vec{n}=\left|\vec{T}^{(n)}\right|\left|\vec{n}\right|\cos\theta$$

i.e. $$\cos\theta=\frac{\vec{T}^{(n)}\cdot\vec{n}}{\left|\vec{T}^{(n)}\right|\left|\vec{n}\right|}=\frac{\frac{3}{\sqrt{6}}\cdot\frac{2}{\sqrt{6}}+\frac{9}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}}+\frac{3}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}}\right)}{\sqrt{\frac{33}{2}}\cdot 1}$$

$$=\frac{12\sqrt{2}}{\sqrt{33}}.$$

Ex.3. *The stress matrix at a point $P(x_i)$ in a material is* $(T_{ij})=\begin{pmatrix}x_1x_3 & x_3^2 & 0\\ x_3^2 & 0 & -x_2\\ 0 & -x_2 & 0\end{pmatrix}$. *Find the stress vector at the point $Q(1,0,-1)$ on the surface $x_1=x_2^2+x_3^2$.*

Ans. Let the surface be $f(x_1,x_2,x_3)=x_1-x_2^2-x_3^2=0$. The outward normal unit vector at any point is given by

$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}, \text{ where } \vec{e}_1 \frac{\partial}{\partial x_1} + \vec{e}_2 \frac{\partial}{\partial x_2} + \vec{e}_3 \frac{\partial}{\partial x_3} \equiv \vec{\nabla}.$$

So, $\vec{\nabla} f = \vec{\nabla}\left(x_1 - x_2^2 - x_3^2\right) = \vec{e}_1 - 2x_2\vec{e}_2 - 2x_3\vec{e}_3$

and $|\vec{\nabla} f| = \sqrt{1 + 4x_2^2 + 4x_3^2}$.

Now, the unit outward normal vector at $Q(1,0,-1)$ on the given surface is $\frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{1}{\sqrt{5}}(\vec{e}_1 + 2\vec{e}_3)$. Therefore

the components of $\vec{n}$ are $n_1 = \frac{1}{\sqrt{5}}, n_2 = 0, n_3 = \frac{2}{\sqrt{5}}$.

The stress vector components at $Q(1,0,-1)$ are as follows:

$$T_1 = T_{11}n_1 + T_{12}n_2 + T_{13}n_3 = -1\cdot\frac{1}{\sqrt{5}} + 1.0 + 0\cdot\frac{2}{\sqrt{5}} = -\frac{1}{\sqrt{5}},$$

$$T_2 = T_{21}n_1 + T_{22}n_2 + T_{23}n_3 = 1\cdot\frac{1}{\sqrt{5}} + 0.0 + 0\cdot\frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}},$$

$$T_3 = T_{31}n_1 + T_{32}n_2 + T_{33}n_3 = 0\cdot\frac{1}{\sqrt{5}} + 0.0 + 0\cdot\frac{2}{\sqrt{5}} = 0.$$

Hence the required stress vector is

$$\vec{T}^{(n)} = T_1\vec{e}_1 + T_2\vec{e}_2 + T_3\vec{e}_3 = -\frac{1}{\sqrt{5}}(\vec{e}_1 - \vec{e}_2)$$

Ex.4. *The state of stress at a point is given by* $(T_{ij}) = \begin{pmatrix} T & aT & bT \\ aT & T & cT \\ bT & cT & T \end{pmatrix}$ *where 'a', 'b', 'c', are constants and T is some stress value. Determine 'a', 'b', 'c' so that the stress vector on a plane normal to* $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ *vanishes.*

Ans. The stress vector at a point is

$$\vec{T}^{(n)} = T_1\vec{e}_1 + T_2\vec{e}_2 + T_3\vec{e}_3$$

where $T_1 = T_{11}n_1 + T_{12}n_2 + T_{13}n_3 = T\cdot\frac{1}{\sqrt{3}} + aT\cdot\frac{1}{\sqrt{3}} + bT\cdot\frac{1}{\sqrt{3}} = \frac{T}{\sqrt{3}}(1 + a + b)$,

$$T_2 = T_{21}n_1 + T_{22}n_2 + T_{23}n_3 = aT\frac{1}{\sqrt{3}} + T\frac{1}{\sqrt{3}} + cT\frac{1}{\sqrt{3}} = \frac{T}{\sqrt{3}}(a+1+c),$$

$$T_3 = T_{31}n_1 + T_{32}n_2 + T_{33}n_3 = bT\frac{1}{\sqrt{3}} + cT\frac{1}{\sqrt{3}} + T\frac{1}{\sqrt{3}} = \frac{T}{\sqrt{3}}(b+c+1).$$

$\therefore \vec{T}^{(n)} = \vec{0}$ implies $T_1 = 0, T_2 = 0, T_3 = 0$. Hence, for $T \neq 0$, we get

$$1+a+b=0$$
$$a+1+c=0$$
$$b+c+1=0$$

which gives $a+b+c = -\frac{3}{2}$ and $a=b=c=-\frac{1}{2}$.

Ex.5. *Determine the Cauchy's stress quatric at a point whose state of stress is* $(T_{ij}) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ *where 'a', 'b', 'c' are all of same sign.*

Ans. The stress quadric is given by $T_{ij}\xi_i\xi_j = 1$

or, $T_{11}\xi_{11}^2 + T_{22}\xi_2^2 + T_{33}\xi_3^2 = 1 \left(\because T_{12} = T_{21} = 0, T_{23} = T_{32} = 0, T_{13} = T_{31} = 0\right)$

or, $\frac{\xi_1^2}{1/a} = \frac{\xi_3^2}{1/b} = \frac{\xi_3^2}{1/c} = 1$

which is the required stress quadric and represents an ellipsoid.

Ex.6. *The stress tensor at a point is* $(T_{ij}) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. *Determine the principal stresses and corresponding principal directions. Also check on the invariance of* $\theta, \theta_2, \theta_3$.

Ans. The principal stresses are given by the characteristic equation

$$\begin{vmatrix} 1-T & 2 & 1 \\ 2 & 1-T & 1 \\ 1 & 1 & 1-T \end{vmatrix} = 0$$

or, $T^3 - 3T^2 - 3T + 1 = 0$

or, $(T+1)(T^2-4T+1)=0$

$\therefore T=-1, 2+\sqrt{3}, 2-\sqrt{3}$

Hence the principal stresses are $T_1=-1, T_2=2+\sqrt{3}, T_3=2-\sqrt{3}$.

Therefore, $\theta=T_1+T_2+T_3=-1+2+\sqrt{3}+2-\sqrt{3}$,

$$\theta=T_1T_2+T_2T_3+T_3T_1=-1\cdot(2+\sqrt{3})+(2+\sqrt{3})(2-\sqrt{3})+(2-\sqrt{3})(-1)=-3,$$

$$\theta_3=T_1T_2T_3=(-1)(2+\sqrt{3})(2-\sqrt{3})=-1.$$

Also, from given stress tensor (T_{ij}), we have

$$\theta=T_{11}+T_{12}+T_{33}=1+1+1=3,$$

$$\theta_2=\begin{vmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{vmatrix}+\begin{vmatrix}T_{22} & T_{23}\\ T_{32} & T_{33}\end{vmatrix}+\begin{vmatrix}T_{33} & T_{31}\\ T_{13} & T_{11}\end{vmatrix}=\begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix}+\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}+\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}=-3,$$

$$\theta_3=\begin{vmatrix}T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{vmatrix}=\begin{vmatrix}1 & 2 & 1\\ 2 & 1 & 1\\ 1 & 1 & 1\end{vmatrix}=1(1-1)-2(2-1)+1(2-1)=-1$$

which implies $\theta, \theta_2, \theta_3$ are invariants.

Let $(l_1, l_2, l_3), (m_1, m_2, m_3), (n_1, n_2, n_3)$ be the direction cosines of the three principal axes corresponding to the three principal stresses $T_1=-1, T_2=2+\sqrt{3}, T_3=2-\sqrt{3}$ respectively. Then we get these sets of homogeneous equations

$$\left.\begin{aligned}2l_1+2l_2+l_3&=0\\ 2l_1+2l_2+l_3&=0\\ l_1+l_2+2l_3&=0\end{aligned}\right\}$$

$$\left.\begin{aligned}(-1-\sqrt{3})m_1+2m_2+m_3&=0\\ 2m_1+(-1-\sqrt{3})m_2+m_3&=0\\ m_1+m_2+(-1-\sqrt{3})m_3&=0\end{aligned}\right\}$$

$$\left.\begin{array}{r}\left(-1+\sqrt{3}\right)n_1+2n_2+n_3=0\\ 2n_1+\left(-1+\sqrt{3}\right)n_2+n_3=0\\ n_1+n_2+\left(-1+\sqrt{3}\right)n_3=0\end{array}\right\}$$

Solving above sets of equations separately we get

$$l_1=\frac{1}{\sqrt{2}},l_2=-\frac{1}{\sqrt{2}},l_3=0;$$

$$m_1=\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{3}}},m_2=\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{3}}},m_3=\sqrt{\frac{1}{3+\sqrt{3}}};$$

$$n_1=-\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{3}}},n_2=-\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{3}}},n_3=\frac{1}{\sqrt{2}}\sqrt{1+\frac{1}{\sqrt{3}}}.$$

Ex.7. *The principal stress at a point are* $T_1=1,T_2=-1,T_3=3$. *If stress at a point is given by* $(T_{ij})=\begin{pmatrix}T_{11}&0&0\\0&1&2\\0&2&T_{33}\end{pmatrix}$.

Find the values of T_{11} *and* T_{33}.

Ans. From the stress invariants we have

$$T_{11}+T_{22}+T_{33}=\theta=T_1+T_2+T_3$$

$$\begin{vmatrix}T_{11}&T_{12}\\T_{21}&T_{22}\end{vmatrix}+\begin{vmatrix}T_{22}&T_{23}\\T_{32}&T_{33}\end{vmatrix}+\begin{vmatrix}T_{33}&T_{31}\\T_{13}&T_{11}\end{vmatrix}=\theta_2=T_1T_2+T_2T_3+T_3T_1$$

$$\begin{vmatrix}T_{11}&T_{12}&T_{13}\\T_{21}&T_{22}&T_{23}\\T_{31}&T_{32}&T_{33}\end{vmatrix}=\theta_3=T_1T_2T_3$$

So, we get $T_{11}+1+T_{33}=1-1+3$ i.e., $T_{11}+T_{33}=2$ .... (i)

$$\begin{vmatrix}T_{11}&0\\0&1\end{vmatrix}+\begin{vmatrix}1&2\\2&T_{33}\end{vmatrix}+\begin{vmatrix}T_{33}&0\\0&T_{11}\end{vmatrix}=1(-1)+(-1)\cdot3+3\cdot1$$

i.e., $T_{11}+T_{33}-4+T_{11}T_{33}=-1$

or, $T_{11}+T_{33}+T_{11}T_{33}=3$ .... (ii)

and $\begin{vmatrix} T_{11} & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & T_{33} \end{vmatrix} = 1\cdot(-1)\cdot 3$

i.e., $T_{11}(T_{33}-4)=-3$

i.e., $T_{11}T_{33}-4T_{11}=-3$ .... (iii)

From (i) and (ii), we get $T_{11}T_{33}=1$ .... (iv)

Using (iv) in (iii), we get $T_{11}=1$ and $T_{33}=1$.

Ex.8. *Find the shearing stress and normal stress on the octahedral plane element through a point whose stress matrix is* $(T_{ij})=\begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$.

Ans. We know that octahedral plane is nothing but a plane whose normal makes equal angles with positive direction of the co-ordinate axes. So, for the octahedral plane, we have the normal $\vec{n}$ has direction $(1,1,1)$, i.e., $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ are the direction cosines of the normal.

Now the components of stress vectors are given by

$$T_1 = T_{11}n_1 + T_{12}n_2 + T_{13}n_3 = 3\cdot\frac{1}{\sqrt{3}}+1\cdot\frac{1}{\sqrt{3}}+1\cdot\frac{1}{\sqrt{3}}=\frac{5}{\sqrt{3}},$$

$$T_2 = T_{21}n_1 + T_{22}n_2 + T_{23}n_3 = 1\cdot\frac{1}{\sqrt{3}}+0\cdot\frac{1}{\sqrt{3}}+2\cdot\frac{1}{\sqrt{3}}=\frac{3}{\sqrt{3}},$$

and $$T_3 = T_{31}n_1 + T_{32}n_2 + T_{33}n_3 = 1\cdot\frac{1}{\sqrt{3}}+2\cdot\frac{1}{\sqrt{3}}+0\cdot\frac{1}{\sqrt{3}}=\frac{3}{\sqrt{3}}.$$

Then the stress vector $\vec{T}^{(n)}$ is

$$\vec{T}^{(n)} = T_1\vec{e}_1 + T_2\vec{e}_2 + T_3\vec{e}_3 = \frac{1}{\sqrt{3}}(5\vec{e}_1+3\vec{e}_2+3\vec{e}_3)$$

Therefore the magnitude of the stress vector is

$$\left|\vec{T}^{(n)}\right| = \frac{1}{\sqrt{3}}\sqrt{5^2+3^3+3^2} = \sqrt{\frac{43}{3}}.$$

Also normal stress at a point is

$$N^{(n)} = T_i^{(n)} = T_{ij}n_in_j$$

$$= T_1n_1 + T_2n_2 + T_3n_3$$

$$= \frac{5}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} = \frac{11}{3}$$

Since $N^{(n)} > 0$, so the normal stress on the plane is tensile shear stress

$$S^{(n)^2} = \left|\vec{T}^{(n)}\right|^2 - \left|N^{(n)}\right|^2 = \frac{43}{3} - \frac{121}{9} = \frac{8}{9}$$

$$\therefore S^{(n)} = \frac{2\sqrt{2}}{3}.$$

Ex.9. *Given the following stress distribution* $(T_{ij}) = \begin{pmatrix} x_2 & -x_3 & 0 \\ -x_3 & 0 & -x_2 \\ 0 & -x_2 & T \end{pmatrix}$ *; find T such that stress distribution is in equilibrium with the body force* $\vec{F} = -g\vec{e}_3$.

Ans. We know that the equation of equlibrium is

$$\rho F_i + T_{ij,j} = 0$$

i.e., $$\rho F_1 + \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = 0$$

$$\rho F_2 + \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} = 0$$

$$\rho F_3 + \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} = 0$$

Here given $\vec{F} = -g\vec{e}_3 \Rightarrow F_3 = -g, F_1 = 0, F_2 = 0$.

So, we have $\frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}+\frac{\partial T_{13}}{\partial x_3}=0 \Rightarrow 0=0,$ .... (i)

$$\frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3}=0 \Rightarrow 0=0, \quad \text{.................. (ii)}$$

and $-\rho g+\frac{\partial T_{31}}{\partial x_1}+\frac{\partial T_{32}}{\partial x_2}+\frac{\partial T_{33}}{\partial x_3}=0 \Rightarrow -\rho g-1+\frac{\partial T}{\partial x_3}=0$ .... (iii)

Obviously, for the given stress distribution the first two equations i.e., (i) and (ii) are identically satisfied. But from (iii) we get

$$\frac{\partial T}{\partial x_3}=1+\rho g$$

Int. $T=x_3(1+\rho g)+f(x_1,x_2)$ .... (iv)

where $f(x_1,x_2)$ is an arbitrary function of x_1,x_2.

Ex.10. *In the absence of body forces, do the stress components*

$$T_{11}=\alpha\left[x_2^2+\beta\left(x_1^2-x_2^2\right)\right], T_{22}=\alpha\left[x_1^2+\beta\left(x_2^2-x_1^2\right)\right], T_{33}=\alpha\beta\left(x_1^2+x_2^2\right)$$

$T_{12}=-2\alpha\beta x_1 x_2, T_{23}=T_{31}=0$ *satisfy the equations of equilibrium?*

Ans. In the absence of the body forces, the equations of equilibrium are

$$\frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}+\frac{\partial T_{13}}{\partial x_3}=0$$

$$\frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3}=0$$

$$\frac{\partial T_{31}}{\partial x_1}+\frac{\partial T_{32}}{\partial x_2}+\frac{\partial T_{33}}{\partial x_3}=0$$

Now, for given stress components,

$$\frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}+\frac{\partial T_{13}}{\partial x_3}=2\alpha\beta x_1-2\alpha\beta x_1=0,$$

$$\frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3}=-2\alpha\beta x_2+2\alpha\beta x_2=0,$$

$$\frac{\partial T_{31}}{\partial x_1}+\frac{\partial T_{32}}{\partial x_2}+\frac{\partial T_{33}}{\partial x_3}=0+0+0=0,$$

which implies that the equations of equilibrium are satisfied.

3.17 Self Assessment Questions :

1. Stress tensor at P are given in approximate units $(T_{ij})=\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 0\\ 2 & 0 & -2\end{pmatrix}$. Find principal stresses and show that principal directions which corresponds to largest and smallest principal stresses are both perpendicular to x_2-axis.

2. Evaluate directly stresses invariants from stress tensor $(T_{ij})=\begin{pmatrix}6 & -3 & 0\\ -3 & 6 & 0\\ 0 & 0 & 8\end{pmatrix}$. Determine principal stresses and show that stress invariants calculated from principal stresses are the same.

3. Stress tensor at a point P are given by $(T_{ij})=\begin{pmatrix}7 & 0 & -2\\ 0 & 5 & 0\\ -2 & 0 & 4\end{pmatrix}$. Determine stress vector on the plane at P whose unit normal is $\frac{2}{3},\frac{-2}{3},\frac{1}{3}$.

4. Show that both of the following have the same principal directions but do not have same principal stresses:

$$\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix} \text{ and } \begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}.$$

5. Stress field is given by $(T_{ij})=\begin{pmatrix}x_1^2x_2 & (1-x_2^2)x_1 & 0\\ (1-x_2^2)x_1 & \frac{(x_2^3-3x_2)}{3} & 0\\ 0 & 0 & 2x_3^2\end{pmatrix}$. Determine (i) body force distribution if

equilibrium equations are to be satisfied, (ii) the principal stresses at the point $P\left(a,0,2\sqrt{a}\right)$.

6. The stress tensor at a point is given by $\begin{pmatrix} 0 & 1 & 2 \\ 1 & b & 1 \\ 2 & 1 & 0 \end{pmatrix}$ where b is a constant. Determine b so that the stress vector on some plane at the point will be zero. Determine also the d.c.'s of the normal to the plane.

7. The state of stress at a point w.r.t. cartesion axes $Ox_1x_2x_3$ is given by $\left(T_{ij}\right)=\begin{pmatrix} 15 & -10 & 0 \\ -10 & 5 & 0 \\ 0 & 0 & 20 \end{pmatrix}$. Determine the stress T'_{ij} for related axes $Ox'_1x'_2x'_3$ for which the transformation matrix is $\left(\alpha_{ij}\right)=\begin{pmatrix} \frac{3}{5} & 0 & \frac{-4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{pmatrix}$.

8. The state of stress at a point is given by $T_{11}=4, T_{22}=2, T_{33}=-2, T_{23}=8, T_{31}=-2, T_{12}=3$. Compute the stress vectors on the planes with unit normals $\left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$ and $\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right)$. Compute the normal and shearing stress on these planes.

9. Is the following stress distribution possible for a body in equilibrium in absence of body forces? $T_{11}=-Ax_1x_2$, $T_{12}=\frac{A}{2}\left(B^2-x_2^2\right)+Cx_3, T_{13}=-Cx_2, T_{22}=T_{33}=T_{23}=0$ where A, B, C are constants.

10. Show that the following state of stress is in equilibrium in absence of body forces: $T_{11}=3x_1^2+3x_2^2-x_3$, $T_{12}=x_3-6x_1x_2-3/4$, $T_{22}=3x_2^2, T_{13}=x_1+x_2-3/2, T_{33}=3x_1+x_2-x_3+\frac{5}{4}, T_{23}=0$.

11. For the following stress distribution $\left(T_{ij}\right)=\begin{pmatrix} x_1+x_2 & T_{12} & 0 \\ T_{12} & x_1-2x_2 & 0 \\ 0 & 0 & x_2 \end{pmatrix}$, find $T_{12}\left(x_1,x_2\right)$ in order that stress distribution is in equilibrium with zero body force and that the stress vector on $x_1=1$ is given by $\vec{T}^{(n)}=\left(1+x_2\right)\vec{e}_1+\left(2-x_2\right)\vec{e}_2$.

3.18 Further Suggested Readings

1. Continuum Mechanics: T.J. Chung, Prentice-Hall.
2. Schaum's Outline of Theory and Problem of Continuum Mechanics: Gedrge R. Mase, McGraw-Hill.
3. Continuum Mechanics: A.J.M. Spencer, Longman.
4. Mathematical Theory of Continuum Mechanics: R.n. Chatterjee, Narosa Publishing House.
5. Foundation of Fluid Mechanics: S.W. Yuan, Prentice-Hall.
6. Fluid Dynamics: J.K. Goyal, K.P. Gupta, Pragati Prakashan.
7. Textbook of Fluid Dynamics: F. Chorlton, CBS Publishers and Distributors.
8. Theory of Elasticity : Yu. Amenzade, Mir Publishers, Moscow.
9. Applied Elasticity : C.T. Wang, McGraw-Hill.

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M.Sc. Course in Applied Mathematics with Oceanology and Computer Programming

PART-I

Paper-V : Group – A : Marks – 50

Module No. - 52
(Mechanics of Continuous Media)

STRUCTURE :

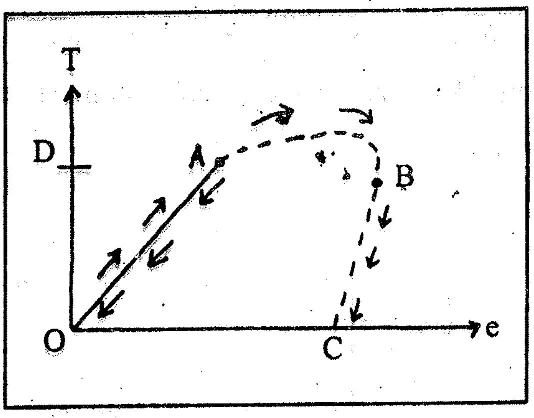
4.1 Introduction :

Elastic Solid : In our universe there are different forms for different materials, and depending on those forms there are different theories of elastic, plastic, viscoelastic bodies, viscous, non-viscous i.e., inviscid fluids etc.

A continuous medium is called *elastic if stress tensor is a continuous function of strain tensor such that stress tensor automatically vanishes when strain tensor becomes zero.* A solid body that consists of this material thus recovers its original shape and size completely whenever all stresses causing the deformation are removed. In this case strain is fully recoverable. *The property by which a continuous body recovers from strain is called elasticity.*

Next we define a linearly elastic solid to be a continuous material (such as metals, concrete, wood) which undergoes very small change of shape when subjected to forces of reasonable magnitude such that every stress component is a linear function of all strain components. It has a natural shape to which it will return whenever all forces causing the deformation are removed provided the forces are not too large. Also it is restricted to the case in which deformation and gradients are small.

To understand the mechanical behaviour of solid, we consider a thin steel rod subjected to a variable tensile stress T. T will produce an extension e. If T is plotted as a function of e, then adjacent figure will be obtained. If T is increased to D so that extension lies within OA, and T is removed then same line OA is retraced, so that there is no permanent deformation or extension and the rod returns to its original length. Then rod exhibits elasticity and greatest stress OD is called elastic limit of the material. If T is increased beyond D such that extension goes from A to B and T is removed, the line BC is retraced, not the curve BAO, so that there is a permanent extension or plastic strain OC after the removal of T. In this case rod does not fully recover its original length completely. The rod exhibits plasticity.



Thus the behaviour of linear elastic material is subjected to the stress within proportional range OA.

4.2 Objectives :

In this module, the students will learn about elastic solid and the general concept of stress-strain relation, wave equation etc.

4.3 Key Words : Elastic solid, Isotrpic media, Hooke's law, Strain energy, Strain energy function, Constitutive equation, Elastic constants, Lame's constant, Bulk modules, Clapeyron's formula, Wave equation.

Hooke's law :

For a linear elastic solid, the strain deformation of a body which gives rise to stresses and the stresses are linear function of infinitesimal strains. So, we can write

$$T_{ij} = B_{ij} + C_{ijkl}e_{kl}$$

Since $T_{ij} = 0$, when all $e_{ij} = 0$, and then $B_{ij} = 0$.

$$\therefore T_{ij} = C_{ijkl}e_{kl}, (i, j, k, l = 1, 2, 3) \text{.......} (1)$$

This relation between stress and strain is known as generalized Hooke's law for linear elastic material.

The coefficients C_{ijkl} are called elastic constants or elastic moduli of the body since they characterise the elastic properties of the body. The elastic solid is said to be elastically non-homogeneous or inhomogeneous if these elastic constants vary from point to point of the medium, and if the elastic constants are the same for all points of the medium then it will be called elastically homogeneous. For an example *mild steel* is homogeneous whereas *reinforced concrete* is non-homogeneous.

Note : We observe that C_{ijkl} form fourth order tensor, known as 'elasticity tensor' and it has $3^4 = 81$ components. Now

$$(C_{ijkl}) = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} & C_{1132} & C_{1113} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} & C_{2232} & C_{2213} & C_{2221} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2131} & C_{2112} & C_{2132} & C_{2113} & C_{2121} \end{pmatrix}$$

Again, since $T_{ij} = T_{ji}$ so from (1) we get $C_{ijkl}e_{kl} = C_{jikl}e_{kl}$ i.e., $C_{ijkl} = C_{jikl}$.

Also $C_{ijkl} = \frac{1}{2}(C_{ijkl} + C_{ijlk}) + \frac{1}{2}(C_{ijkl} - C_{ijlk}) = C'_{ijkl} + C''_{ijkl}$, say

where $C'_{ijkl} = C'_{ijlk}$ and $C''_{ijkl} = -C''_{ijlk}$,

Which gives $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk}$ for symetry of T_{ij} and e_{kl}.

Hence C_{ijkl} has 36 components instead of 81. The matrix of the coefficients C_{ijkl} takes the form

$$\begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ C_{2211} & C_{2212} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} \end{pmatrix}$$

Equation (1) is generatlized Hooke's law or stress-strain relation or constitutive equation of linearly elastic solid.

4.5 Strain Energy :

Physically when an elastic body is under the action of external surface forces, the body deform and external surface forces that act on the body do a certain amount of work. The work done in straining such an elastic body from the configuration of unstrained state to the present state by surface force is transformed completely into the potential energy stored in the body. This potential energy is due to deformation or strain only. It is called strain energy of the elastic body.

4.6 Existence of Strain Energy Function :

The principle of conservation of energy i.e., first law of thermodynamics gives

$$\rho \frac{de}{dt} = T_{ij} d_{ij} - q_{i,i} + \rho h \text{ .... (2)}$$

where e = internal energy per unit mass,

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

q_i = the flux heat by conduction per unit area per unit time,

h = rate per unit mass at which heat energy by radiation is produced from internal sources.

For linear elastic solid it is assumed that heat conduction is neglected and that heat energy is produced entirely by internal sources only. So, $q_i = 0$. Then from (2) we get,

$$\rho \frac{de}{dt} = T_{ij} d_{ij} + \rho h \text{ .... (3)}$$

Now, for small strains we have

$$d_{ij} = \dot{e}_{ij}$$

and if Q_1 be the quantity of heat per unit mass produced by internal sources at time t, then

$$h = \frac{dQ_1}{dt}.$$

Hence (3) reduces to,

$$T_{ij}\dot{e}_{ij} = \rho\left(\dot{e} - \dot{Q}_1\right) \text{ .... (4)}$$

Now we introduce

$$U = \rho_0 e \text{ and } Q = \rho_0 Q_1 \text{ .... (5)}$$

Here, U is the internal energy per unit volume of the unstrained state of the body and Q is the quantity of heat produced from internal sources per unit volume of the unstrained state. So, (5) takes the form

$$T_{ij}\dot{e}_{ij} = \frac{\rho}{\rho_0}\left(\dot{U} - \dot{Q}\right) \text{ .... (6)}$$

But $\frac{\rho}{\rho_0} = \frac{dV_0}{dV} = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \left|\frac{\partial x_i}{\partial X_j}\right| = \left|u_{i,j} + \delta_{ij}\right| = u_{i,i} + 1$. (for small displacements)

For small displacement gradients we have $\frac{\rho}{\rho_0} = 1$, i,e., $\rho = \rho_0$. Then (6) reduces into

$$T_{ij}\dot{e}_{ij} = \dot{U} - \dot{Q} \text{ .... (7)}$$

Hence first law of thermodynamics gives the above result.

Again, from the second law of thermodynamics, we have

$$\dot{Q} = \frac{\partial Q}{\partial t} = T\frac{\partial S}{\partial t} \text{ .... (8)}$$

where S is the entropy and T is the temperature per unit volume.

If the change of state from one configuration to another takes place adiabatically, then the change takes place so rapidly that there is no time for the heat generated to the dissipated. This is referred as adiabatic process and consequently $\dot{Q} = 0$. Hence (7) gives

$$T_{ij}\dot{e}_{ij} = \dot{U}$$

or, $\quad T_{ij}de_{ij} = dU$ .... (9)

which shows that L.H.S. of (9) is an exact differential as R.H.S. is. So, there exist a function W such that

$$T_{ij}de_{ij} = dW \text{ .... (10)}$$

where W is the internal energy of the body.

If the change of state takes place isothermally in which the change is so slow that heat generated has time enough to be dissipated so that temperature of the body remains constants and the body is in continued equilibrium of temperature with surrounding bodies, and consequently

$$\frac{\partial T}{\partial t} = \dot{T} = 0$$

Then (8) becomes

$$\dot{Q} = \dot{T}\frac{\partial S}{\partial t} + S\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}(TS) \text{ .... } (11)$$

Eliminating $\dot{Q}$ from (7) and (11), we get

$$T_{ij}\dot{e}_{ij} = \frac{\partial}{\partial t}[U - TS] = \frac{\partial F}{\partial t} = \dot{F}, \text{ say, ... } (12)$$

where $F = U - TS$, is the *Helmholtz's* free energy.

$$\therefore T_{ij}de_{ij} = dF \text{ (using (12)) .... } (13)$$

which shows that L.H.S. is an exact differential. Therefore, there exist a function W such that

$$T_{ij}de_{ij} = dW \text{ .... } (14)$$

Here, W represents the Helmholtz's free energy per unit volume of the elastic medium.

Now we introduce the following notations to avoid double sum:

$T_{11} = T_1, T_{22} = T_2, T_{33} = T_3$

$T_{23} = T_{32} = T_4,$

$T_{31} = T_{13} = T_5,$

$T_{12} = T_{21} = T_6,$

and $e_{11} = e_1, e_{22} = e_2, e_{33} = e_3,$

$2e_{23} = 2e_{32} = e_4,$

$2e_{31} = 2e_{13} = e_5,$

$2e_{12} = 2e_{21} = e_6.$

Then, we have,

$$T_{ij}de_{ij} = T_i de_i \, (i = 1, 2, ..., 6)$$

Hence, from (10) and (14), whether change of state is isothermal or adiabatic, there exist a function W such that

$$T_i de_i = dW \left(i = 1, 2, ..., 6 \right) \text{......... (15)}$$

Now $T_i de_i = T_{ij} de_{ij}$ re presents the work done per unit volume at a point by all surface forces, and therefore dW represents the work done per unit volume. If W is a function of independent variables $e_1, e_2, ..., e_6$, then

$$dW = \frac{\partial W}{\partial e_i} de_i \text{ .... (16)}$$

$$\therefore T_i de_i = \frac{\partial W}{\partial e_i} de_i$$

$$\Rightarrow T_i = \frac{\partial W}{\partial e_i} \left(i = 1, 2, ..., 6 \right) \text{ .... (17)}$$

Thus, for the both processes (isothermal and adiabatic) there exist a function W with the property $T_i = \frac{\partial W}{\partial e_i}$.

This function $W = W\left(e_1, e_2, ..., e_6\right)$ is called the ***stress potential*** or strain energy function for an unit volume of the elastic body, as it is the potential energy per unit volume stored up in the body by strain. Equation (16) gives stress components in terms of partial derivatives of strain energy w.r.t. corresponding strain components.

4.7 Istropic Elastic Solid :

A linearly elastic solid is known as to be isotropic *if it has the same elastic symmetry in all directions.* It menas the strain energy W must be invariant under all orthogonal transformations of co-ordinate axes. That is, in other words, W is independent of the orientation of co-ordinate axes and hence W must be expressed in terms of invariants of strain tensor.

Constitutive Equation :

The generalized Hooke's law for linearly elastic material is

$$T_{ij} = a_{ijkl} e_{kl} \left(i = 1, 2, 3; j = 1, 2, 3 \right)$$

$$= \frac{1}{2}\left(a_{ijkl} + a_{ijlk}\right) + \frac{1}{2}\left(a_{ijkl} - a_{ijlk}\right)$$

$$= b_{ijkl} + c_{ijkl}, \text{ say}$$

where $b_{ijkl} = \frac{1}{2}\left(a_{ijkl} + a_{ijlk}\right) = b_{ijlk}$

and $c_{ijkl} = \frac{1}{2}\left(a_{ijkl} - a_{ijlk}\right) = -c_{ijlk}$

$\therefore T_{ij} = b_{ijkl}e_{kl} + c_{ijkl}e_{kl}$

Also $c_{ijkl}e_{kl} = c_{ijlk}e_{lk}$ (interchanging dummy suffixes)

$= -c_{ijkl}e_{lk}$

$= -c_{ijkl}e_{kl}$ ($\because e_{kl}$ is symmetric)

$\Rightarrow c_{ijkl}e_{kl} = 0.$

Hence $T_{ij} = b_{ijkl}e_{kl}$

where $b_{ijkl} = b_{ijlk}$

Since T_{ij} and e_{kl} form second order tensor, and hence b_{ijkl} form a tensor of order four. For an isotropic elastic medium the elastic constants b_{ijkl} remains the same under all orthogonal transformation of co-ordinates axes. Thus for isotropic body b_{ijkl} must be an isotropic tensor of order four. Therefore, we can write

$$b_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \text{ .... (18)}$$

where λ, μ, γ are constants.

or, $\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} = \lambda\delta_{ij}\delta_{lk} + \mu\delta_{il}\delta_{jk} + \gamma\delta_{ik}\delta_{jl}$

$$(\because b_{ijkl} = b_{ijlk})$$

or, $(\mu-\gamma)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = 0$

which is true for all values of *i, j, k, l*.

Setting $i = 1, k = 1, j = 2, l = 2$ then above relation becomes

$(\mu-\gamma)(\delta_{11}\delta_{22} - \delta_{12}\delta_{21}) = 0$

or, $(\mu-\gamma) = 0$

i.e., $\mu = \gamma$

Therefore, (18) takes the form

$$b_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \text{ .... (19)}$$

Now the generalized Hooke's law reduces to

$$T_{ij} = [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]e_{kl}$$

$$= \lambda\delta_{ij}e_{kk} + \mu(e_{il}\delta_{jl} + e_{kl}\delta_{jk})$$

$$= \lambda\delta_{ij}e_{kk} + \mu(e_{ij} + e_{ji})$$

$$\therefore T_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij} \quad \text{...........} (20)$$

where $\theta = e_{kk}$, is the first invariant.

Equations (20) are the *constitutive equations* or *stress-strain relation* for an isotropic linearly elastic body.

The number of elastic constants, in (20), are only two, namely λ and μ. Now the strain energy W is given by

$$W = \frac{1}{2} T_{ij} e_{ij} \left(\because dW = T_{ij} de_{ij}\right)$$

$$= \frac{1}{2}\left(\lambda\theta\delta_{ij} + 2\mu e_{ij}\right) e_{ij}$$

$$= \frac{1}{2}\lambda\theta^2 + \mu e_{ij} e_{ij}$$

$$= \frac{1}{2}\lambda\theta^2 + \mu\left(e_{11}^2 + e_{22}^2 + e_{33}^2 + 2e_{12}^2 + 2e_{23}^2 + 2e_{31}^2\right)$$

Thus strain energy W is a positive definite form in the strain e_{ij} taking positive values only.

Theorem : Principal directions of strain at each point of a linearly elastic isotropic body are coincident with the principal directions of stress.

Proof. : Let us consider the principal directions of strain at a point of the medium as co-ordinates axes. So, if e_{ij} and T_{ij} be the strain and stress tensors at that point, then we have

$$e_{12} = 0, e_{23} = 0, e_{31} = 0.$$

Also from the stress-strain relation for the linearly elastic isotropic body we have the constitutive equation as

$$T_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}$$

$$\therefore T_{12} = \lambda\theta\delta_{12} + 2\mu e_{12} = 2\mu e_{12} = 0$$

$$\therefore T_{23} = \lambda\theta\delta_{23} + 2\mu e_{23} = 2\mu e_{23} = 0$$

and $T_{31} = \lambda\theta\delta_{31} + 2\mu e_{31} = 2\mu e_{31} = 0$

which implies that the co-ordinate axes must be along the principal directions of strain are coincides with the principal directions of stress for an isotropic body, that means for an isotropic body there is no distinction made between principal direction of strain and those of stress. Both are referred as principal directions.

Note 1. Steel, aluminium, glass are examples of isotropic material.

Note 2. A material elastically symmetry w.r.t. a plane with 13 elastic coefficients as $c_{11}, c_{12}, c_{13}, c_{16}, c_{22}, c_{23}, c_{26}, c_{33}, c_{36}, c_{44}, c_{45}, c_{66}$, is called *monotropic* material.

Note 3. A material having three mutually perpendicular planes of elastic symmetry is said to be *orthotropic*. Wood is an example of an orthotropic elastic material and in this case W is given by

$$W = \frac{1}{2}\left[c_{11}e_1^2 + c_{22}e_2^2 + c_{33}e_3^2 + c_{44}e_4^2 + c_{55}e_5^2 + c_{66}e_6^2 + 2c_{12}e_1e_2 \right.$$

$$\left. +2c_{13}e_1e_3 + 2c_{23}e_2e_3\right]$$

which contains only 9 non-zero elastic constants instead of 13 in case of monotropic material.

Note 4. A material with 21 elastic constants $c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, c_{22}, c_{23}, c_{24}, c_{25}, c_{26}, c_{33}, c_{34}, c_{35}, c_{36}, c_{44}, c_{45}, c_{46}, c_{55}, c_{56}, c_{66}$ is called *anistropic* linerly elastic material. For an example of anisotropic body is provided by crystal.

4.8 Basic Elastic Constants : for Isotropic Solid :

For the isotropic linear elastic material, the constitutive equation, is

$$T_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}$$

where λ and μ are two elastic constants, known as *Lame's constants*.

$$\therefore T_{ii} = \lambda\theta\delta_{ii} + 2\mu e_{ii}$$

$$= 3\lambda\theta + 2\mu\theta \; (\because \theta = e_{ii})$$

$$= (3\lambda + 2\mu)\theta$$

or, $\quad \Theta = (3\lambda + 2\mu)\theta$ where $T_{ii} = \Theta$.

$$\therefore \theta = \frac{\Theta}{3\lambda + 2\mu} \quad \ldots\ldots\ldots\ldots (21)$$

Hence, $\quad T_{ij} = \dfrac{\lambda\Theta}{3\lambda + 2\mu}\delta_{ij} + 2\mu e_{ij}$

or, $\quad e_{ij} = \dfrac{T_{ij}}{2\mu} - \dfrac{\lambda\Theta\delta_{ij}}{2\mu(3\lambda + 2\mu)} \quad \ldots\ldots\ldots (22)$

which is the strain-stress relation and known as the *inverston of Hooke's law*.

Now, in particular, for $i=1, j=1$, we have

$$e_{11} = \frac{T_{11}}{2\mu} - \frac{\lambda\Theta}{2\mu(3\lambda+2\mu)}$$

$$= \frac{T_{11}}{2\mu}\left[1 - \frac{\lambda}{3\lambda+2\mu}\right] - \frac{\lambda(T_{22}+T_{33})}{2\mu(3\lambda+2\mu)}$$

$$= \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} \cdot T_{11} - \frac{\lambda}{2\mu(3\lambda+2\mu)}(T_{22}+T_{33}) \text{ .... (23)}$$

Let us consider the situation T_{11} = Constant = T, $T_{22} = T_{33} = T_{23} = T_{31} = T_{12} = 0$. This state of stress is possible in an elastic right circular cylinder the axis of which is parallel to x_1 axis and subjected to an uniform longitudinal axial tensile loading to both of its ends. Also, the above state of stress satisfies the equilibrium equations in absence of body forces at every point in the interior of the cylindrical elastic medium and also satisfies the stress-free boundary condition on its laterial surface.

Using (23) in (22), then we get,

$$\left.\begin{aligned} & e_{11} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} T, e_{22} = e_{33} = -\frac{\lambda}{2\mu(3\lambda+2\mu)} \cdot T, \\ & e_{23} = 0, e_{31} = 0, e_{12} = 0. \end{aligned}\right\} \text{ .... (24)}$$

Now, the ratio of the tensile stress T_{11} to the longitudinal extension e_{11}, i.e.,

$$\frac{T_{11}}{e_{11}} = \frac{T}{e_{11}} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \text{Constant.}$$

This ratio is called *Young's modules* or *modulus of elasticity*, and is denoted by E. Hence, by definition we have

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \text{ .... (25)}$$

Also, we consider another ratio of lateral contraction to the longitudinal extension, i.e.,

$$-\frac{e_{22}}{e_{11}} = \frac{\lambda}{2(\lambda+\mu)} = \text{Constant.}$$

This ratio is called *Poisson's ratio*, and is denoted by σ.

Hence

$$\sigma = \frac{\lambda}{2(\lambda+\mu)} \text{ .... (26)}$$

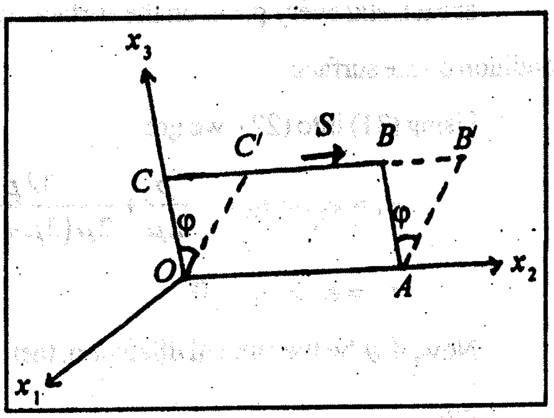
Now, from (25) and (26) we get

$$\left.\begin{aligned} \lambda &= \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \\ \text{and} \quad \mu &= \frac{E}{2(1+\sigma)} \end{aligned}\right\} \text{............ (27)}$$

Next we consider the state of stress

$$T_{11} = T_{22} = T_{33} = T_{12} = T_{31} = 0,\ T_{23} = S = \text{Constant} \text{........ (28)}$$

This state of stress is possible in a deformed long rectangular parallelopiped of square cross-section *OABC* which is sheared in the plane containing *OA* and *OC* by a shearing stress of magnitude *S* acting per unit area on the side *CB*. Now the stress *S* would tend to slide the planes of the material originally perpendicular to *OC*, the x_3-axis, in a direction parallel to *OA*, the x_2-axis so that the right angle between *OA* and *OC* will diminished by an angle φ. This state of stress satisfies the equations of equilibrium in the absence of body force at every point in the interior and the boundary condition on the surface.



Now from (22), using (28), we get

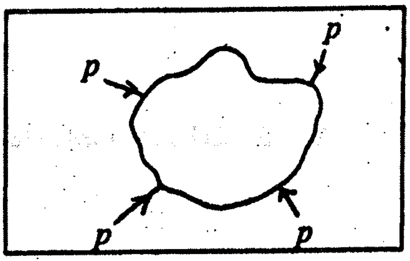
$$e_{23} = \frac{T_{23}}{2\mu} = \frac{S}{2\mu},\ e_{11} = e_{22} = e_{33} = e_{31} = e_{12} = 0 \text{........... (29)}$$

Also, from the definition

$$2e_{23} = \varphi = \text{Change in angle}$$

$$\therefore \frac{S}{\varphi} = \frac{S}{2e_{23}} = \mu \text{........... (30)}$$

This ratio is known as *shear modulus* or *modulus of rigidity*, and it is identical with Lame's constant μ.



Finally consider an elastic body of arbitrary shape subjected to a hydrostatic stress *p* distributed over its surface. Also the hydrostatic stress diminishes the volume of the body. This state of stress possible in such a deformed body is given by

$$T_{11} = T_{22} = T_{33} = -p = \text{constant.}$$
$$T_{23} = T_{31} = T_{12} = 0$$

The state of stress satisfies equilibrium equation in the interior of body. If $T_i^{(n)}$ be stress vector acting on the surface with normal n_i, then

$$T_i^{(n)} = -pn_i \left(i = 1,2,3\right)$$

and hence

$$T_{ij}n_i = -pn_i \text{ .... (32)}$$

at each and every point on the surface. It is obvious that state of stress in (31) also satisfies the boundary condition on the surface.

Using (31) into (22), we get

$$\left.\begin{aligned} e_{11} = e_{22} = e_{33} &= -\frac{p}{2\mu} + \frac{3\lambda p}{2\mu(3\mu+2\mu)} = -\frac{p}{3\lambda+2\mu} = \text{constant.} \\ e_{12} = e_{23} = e_{31} &= 0. \end{aligned}\right\} \text{ .... (33)}$$

Now, if θ be the cubical dilatation, then the decrease in volume per unit volume is $(-\theta)$.

So,

$$-\theta = -e_{ii} = -\left(e_{11} + e_{22} + e_{33}\right)$$
$$= \frac{3p}{3\lambda + 2\mu}.$$

Thus the ratio of the hydrostatic stress to the decrease in volume per unit volume, i.e.,

$$\frac{p}{-\theta} = \frac{3\lambda + 2\mu}{3} = \lambda + \frac{2\mu}{3} = \text{constant.}$$

This constant is known as *bulk modulus* or *modulus of compression*, and is denoted by K. So,

$$K = \lambda + \frac{2\mu}{3} \text{ .... (34)}$$

Also K can be express in terms of E and σ as

$$K = \frac{E}{3(1-2\sigma)} \text{ .... (35)}$$

If $K>0$, $E>0$, then we must have from (35)

$$0<\sigma<\frac{1}{2}$$

Again, since $0<\sigma<\frac{1}{2}$ and $E>0$, then we have $\lambda>0, \mu>0$.

For most general materials σ does not deviate from $\frac{1}{3}$ and if the material is incompressible

$$\theta=0, K\to\infty, \sigma=\frac{1}{2}, \mu=\frac{E}{3} \text{......... (36)}$$

For some solids and rocks $\lambda=\mu$ and in this case

$$K=\frac{5\mu}{3}, E=\frac{5\mu}{2}, \gamma=\frac{1}{4}.$$

Now, from (25) and (26), we have

$$\frac{1+\sigma}{E}=\frac{1}{2\mu}$$

and $$\frac{\sigma}{E}=\frac{\lambda}{2\mu(3\lambda+2\mu)}.$$

Hence the stress-strain relation becomes

$$e_{ij}=\frac{1+\sigma}{E}T_{ij}-\frac{\sigma\Theta\delta_{ij}}{E} \text{ .... (37)}$$

and cubical dilatation θ is

$$\theta=e_{ii}=\frac{1+\sigma}{E}T_{ii}-\frac{3\sigma}{E}\Theta$$

$$=\frac{1+\sigma}{E}\cdot\Theta-\frac{3\sigma}{E}\cdot\Theta$$

$$=\frac{1-2\sigma}{E}\cdot\Theta$$

or, $$\theta=\frac{\Theta}{3K} \text{ (using 35)}$$

$$\therefore \frac{\Theta}{3}=K\theta \text{ .... (38)}$$

which shows that dilatation depends on average normal stress.

Corollary : Express strain energy function W in terms of stresses only.

Ans. From Clapeyron's formula for strain energy function we have

$$W = \frac{1}{2}T_{ij}e_{ij}.$$

Hence, from (37), we get

$$W = \frac{1}{2}T_{ij}\left[\frac{1+\sigma}{E}T_{ij} - \frac{\sigma}{E}\Theta\delta_{ij}\right]$$

$$= \frac{1+\sigma}{2E}T_{ij}T_{ij} - \frac{\sigma\Theta^2}{2E}$$

$$= \frac{(1+\sigma)}{2E}\left[T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{31}^2 + 2T_{12}^2 + 2T_{23}^2\right] - \frac{\sigma}{2E}\left[T_{11} + T_{22} + T_{33}\right]^2.$$

4.9 Beltrami-Michell Compatibility Equations :

The compatibility equations for the strain components are

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$$

Combining above equation linearly by setting $l=k$ and summing over k, then we get

$$e_{ij,kk} + e_{kk,ij} - e_{ik,jk} - e_{jk,ik} = 0$$

Now, the strain-stress law for isotropic elastic body is

$$e_{ij} = \frac{1+\sigma}{E}T_{ij} - \frac{\sigma}{E}\Theta\delta_{ij} = \frac{1+\sigma}{E}\left(T_{ij} - \frac{\sigma}{1+\sigma}\Theta\delta_{ij}\right)$$

$$\therefore T_{ij,kk} + T_{kk,ij} - T_{ik,jk} - T_{jk,ik} = \frac{\sigma}{1+\sigma}\left(\delta_{ij}\Theta_{,kk} + \delta_{kk}\Theta_{,ij} - \delta_{ik}\Theta_{,jk} - \delta_{jk}\Theta_{,ik}\right)$$

or, $$\nabla^2 T_{ij} + \Theta_{,ij} - \frac{\sigma}{1+\sigma}\delta_{ij}\nabla^2\Theta - \frac{3\sigma}{1+\sigma}\Theta_{,ij} + \frac{\sigma}{1+\sigma}\Theta_{,ji} + \frac{\sigma}{1+\sigma}\Theta_{,ij} = T_{ik,jk} + T_{jk,ik}$$

or, $$\nabla^2 T_{ij} + \frac{\sigma}{1+\sigma}\Theta_{,ij} - \frac{\sigma}{1+\sigma}\delta_{ij}\nabla^2\Theta = T_{ik,jk} + T_{jk,ik} \quad \text{............. (39)}$$

Again the equations of equilibrium are

$$T_{ik,k} + \rho F_i = 0$$

$$\therefore T_{ik,kj} = -\rho F_{i,j}$$

Similarly,

$$\therefore T_{jk,ik} = -\rho F_{j,i}$$

Using above results in (39), we get

$$\nabla^2 T_{ij} + \frac{1}{1+\sigma}\Theta,_{ij} - \frac{\sigma}{1+\sigma}\delta_{ij}\nabla^2\Theta = -\rho\left(F_{i,j} + F_{j,i}\right) \ldots\ldots (40)$$

Setting $j=i$ and summing over common index i then we have

$$\nabla^2\Theta + \frac{1}{1+\sigma}\nabla^2\Theta - \frac{3\sigma}{1+\sigma}\nabla^2\Theta = -2\rho\, div.\vec{F}$$

Or, $$\nabla^2\Theta = -2\left(\frac{1+\sigma}{1-\sigma}\right)\rho\ div.\vec{F} \ldots\ldots\ldots (41)$$

Using (41) into (40) then we get

$$\nabla^2 T_{ij} + \frac{1}{1+\sigma}\Theta,_{ij} = -\frac{2\sigma}{1-\sigma}\delta_{ij}\ \rho\ div.\vec{F} - \rho\left(F_{i,j} + F_{j,i}\right) \ldots\ldots\ldots\ldots (42)$$

These are six independent equations which are known as Beltrami-Michell Compatibility equation for stresses.

Note 1. Beltrami-Michell compatibility equations for stresses given by (42) are suitable only for isotropic elastic body whereas Saint-Venant's compatibility equations for strain are suitable for any body.

Note 2. Beltrami-Michell compatibility equation sometime known as First Fundamental Boundary Value Problem.

4.10 Wave equation : Navier's equations of motion :

The strain components are given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and strain-stress relation is

$$T_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij} \text{ where } \theta = e_{11} + e_{22} + e_{33} = \frac{\partial u_k}{\partial x_k}$$

$$= \lambda\theta\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \lambda\theta\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right)$$

$$\therefore T_{ij,j} = \lambda\delta_{ij}\theta,_j + \mu\left(u_{i,jj} + u_{j,ij}\right)$$

$$= \lambda\theta_{,i} + \mu\left(u_{i,jj} + u_{j,ij}\right)$$

$$= \lambda\theta_{,i} + \mu\nabla^2 u_i + \mu\left(u_{j,j}\right)_{,i} \qquad \left(\because \frac{\partial^2 u_j}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)\right)$$

$$= \lambda\theta_{,i} + \mu\nabla^2 u_i + \mu\theta_{,i}$$

$$= (\lambda + \mu)\theta_{,i} + \mu\nabla^2 u_i$$

Substituting these into the equation of motion

$$T_{ij,j} + \rho F_i = \rho\ddot{u}_i$$

Then we get

$$(\lambda + \mu)\theta_{,i} + \mu\nabla^2 u_i + \rho F_i = \rho\ddot{u}_i \text{............} (43)$$

which is known as Navier's equation of motion.

When body forces are absent i.e., when $F_i = 0$ then Nvier's equation of motion reduces to

$$(\lambda + \mu)\frac{\partial\theta}{\partial x_i} + \mu\nabla^2 u_i = \rho\ddot{u}_i \text{..........} (44)$$

When the motion is irrotational then

$$u_i = \phi_{,i}$$

$$\therefore \theta = u_{i,i} = \phi_{,ii} = \nabla^2\phi$$

and $\theta_{,i} = \left(\nabla^2\phi\right)_{,i} = \nabla^2\left(\phi_{,i}\right) = \nabla^2 u_i$.

So, the equation of motion reduces to

$$(\lambda + \mu)\nabla^2 u_i + \mu\nabla^2 u_i = \rho\ddot{u}_i$$

or, $(\lambda + 2\mu)\nabla^2 u_i = \rho\ddot{u}_i$

or, $\dfrac{\partial^2 u_i}{\partial t^2} = c_1^2\nabla^2 u_i$ .... (45)

where $c_1 = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$

which is a wave equation for displacement called irrotational waves propagating with the velocity $c_1 = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$.

When the motion is isochoric, then

$$\theta = e_{11} + e_{22} + e_{33} = 0$$

and hence equation (44) reduces to

$$\mu\nabla^2 u_i = \rho\ddot{u}_i$$

or, $$\frac{\partial^2 u_i}{\partial t^2} = c_2^2\nabla^2 u_i \text{ .... (46)}$$

where $\frac{\partial^2 u_i}{\partial t^2} = c_2^2\nabla^2 u_i$.

This equation represents a wave equation for displacement called equivolumnal waves, propagating with the velocity $c_2 = \sqrt{\frac{\mu}{\rho}}$.

4.11 Unit Summary :

In this module we have discussed about Hooke's law, strain energy function, relations among Elastic constants, wave equation etc. which has covered all portions of stress-strain.

4.12 Worked out Examples

Ex.-1 *If the principal directions of stress are coincident with the principal directions of strain and if the relations between principal stresses T_i and principal strains e_i are $T_i = \lambda\theta + 2\mu e_i$, then prove that stress-strain relations for any system of axes are*

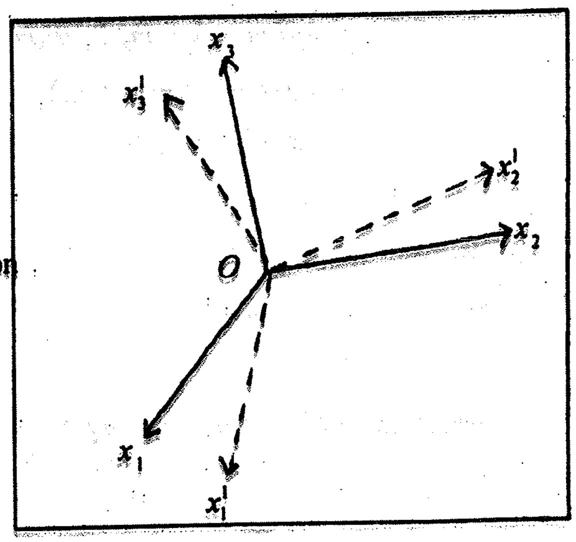
$$T_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}.$$

Ans. Let Ox_1, Ox_2, Ox_3 be a set of rectangular axes along the principal directions of stress and strain. Then

$T_{11} = T_1, T_{22} = T_2, T_{33} = T_3, T_{12} = T_{23} = T_{31} = 0;$

$e_{11} = e_1, e_{22} = e_2, e_{33} = e_3, e_{12} = e_{23} = e_{31} = 0.$

Let Ox_1', Ox_2', Ox_3' be another set of axes with direction cosines $l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3$; respectively.



By the law of transformation of stress components we have

$$T_{11}' = T_{ij}l_il_j = T_{11}l_1^2 + T_{22}l_2^2 + T_{33}l_3^2 = T_1l_1^2 + T_2l_2^2 + T_3l_3^2$$

$$T'_{12} = T_{ij}l_i m_j = T_{11}l_1m_1 + T_{22}l_2m_2 + T_{33}l_3m_3$$
$$= T_1l_1m_1 + T_2l_2m_2 + T_3l_3m_3$$

Similarly from law of transformation of strain components we get

$$e'_{11} = e_1l_1^2 + e_2l_2^2 + e_3l_3^2,$$
$$e'_{12} = e_1l_1m_1 + e_2l_2m_2 + e_3l_3m_3$$

$$\therefore T'_{11} - \lambda\theta - 2\mu e'_{11}$$
$$= T_1l_1^2 + T_2l_2^2 + T_3l_3^2 - \lambda\theta(e_1 + e_2 + e_3) - 2\mu(e_1l_1^2 + e_2l_2^2 + e_3l_3^2)$$
$$= (T_1 - 2\mu e_1)l_1^2 + (T_2 - 2\mu e_2)l_2^2 + (T_3 - 2\mu e_3)l_3^2 - \lambda\theta$$
$$= \lambda\theta l_1^2 + \lambda\theta l_2^2 + \lambda\theta l_3^2 - \lambda\theta \text{ (using given condition)}$$
$$= \lambda\theta(l_1^2 + l_2^2 + l_3^2) - \lambda\theta$$
$$= 0$$

Hence, $T'_{11} = \lambda\theta + 2\mu e'_{11}$

Similarly, $T'_{22} = \lambda\theta + 2\mu e'_{22}$ .... (i)

$T'_{33} = \lambda\theta + 2\mu e'_{33}$.

Again,

$$T'_{12} - 2\mu e'_{12} = T_1l_1m_1 + T_2l_2m_2 + T_3l_3m_3 - 2\mu(e_1l_1m_1 + e_2l_2m_2 + e_3l_3m_3)$$
$$= (T_1 - 2\mu e_1)l_1m_1 + (T_2 - 2\mu e_2)l_2m_2 + (T_3 - 2\mu e_3)l_3m_3$$
$$= \lambda\theta l_1m_1 + \lambda\theta l_2m_2 + \lambda\theta l_3m_3 \text{ (using given condition)}$$
$$= \lambda\theta(l_1m_1 + l_2m_2 + l_3m_3)$$
$$= \lambda\theta \cdot 0$$
$$= 0.$$

Hence, $T'_{12} = 2\mu e'_{12}$

Similarly, $T'_{23} = 2\mu e'_{23}$ .... (ii)

$T'_{31} = 2\mu e'_{31}$

Hence, combining (i) and (ii), we get

$$T'_{ij} = \lambda\theta\delta_{ij} + 2\mu e'_{ij}$$

Ex-2. *If the stress invariant Θ and strain invariant θ are connected by the relations $\Theta = 3K\theta$ and $T_{ij} = 2\mu e_{ij}\,(i \neq j)$ for all pairs of rectangular axes ox_i and ox_j; then prove that*

$$T_{11} = \lambda\theta + 2\mu e_{11}, T_{22} = \lambda\theta + 2\mu e_{22}, T_{33} = \lambda\theta + 2\mu e_{33}$$

for any system of axes.

Ans. Let us assume that the co-ordinate axes Ox_i along the principal directions of strain. Then we must have

$$e_{12} = e_{23} = e_{31} = 0.$$

Also, given that,

$$T_{ij} = 2\mu e_{ij}\,(i \neq j)$$

$$\therefore T_{12} = 2\mu e_{12} = 0,$$

$$T_{23} = 2\mu e_{23} = 0,$$

and $T_{31} = 2\mu e_{31} = 0.$

Hence the axes Ox_i must be along the directions of principal stress also. If ox_1', ox_2', ox_3', be the another system of co-ordinate axes with direction cosines $l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3$ respectively. Now from the law of transformation of stress and strain components we have

$$T_{12}' = T_{ij} l_i m_j = T_{11} l_1 m_1 + T_{22} l_2 m_2 + T_{33} l_3 m_3$$

and $e_{12}' = e_{ij} l_i m_j = e_{11} l_1 m_1 + e_{22} l_2 m_2 + e_{33} l_3 m_3.$

Now from the given relation $T_{ij}' = 2\mu e_{ij}'\,(i \neq j)$ we get

$$T_{12}' - 2\mu e_{12}' = 0$$

Hence

$$T_{11} l_1 m_1 + T_{22} l_2 m_2 + T_{33} l_3 m_3 - 2\mu\left(e_{11} l_1 m_1 + e_{22} l_2 m_2 + e_{33} l_3 m_3\right) = 0$$

or, $$\left(T_{11} - 2\mu e_{11}\right) l_1 m_1 + \left(T_{22} - 2\mu e_{22}\right) l_2 m_2 + \left(T_{33} - 2\mu e_{33}\right) l_3 m_3 = 0$$

Again, since $l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$, therefore

$$\left(T_{11} - 2\mu e_{11}\right) = \left(T_{22} - 2\mu e_{22}\right) = \left(T_{33} - 2\mu e_{33}\right)$$

or, $$\frac{T_{11} - 2\mu e_{11}}{1} = \frac{T_{22} - 2\mu e_{22}}{1} = \frac{T_{33} - 2\mu e_{33}}{1} = \frac{T_{11} + T_{22} + T_{33} - 2\mu\left(e_{11} + e_{22} + e_{33}\right)}{3}$$

$$= \frac{\Theta - 2\mu\theta}{3} = \frac{3K\theta - 2\mu\theta}{3}$$

$$= \theta\left(K - \frac{2\mu}{3}\right)$$

$$= \theta\left(\lambda + \frac{2\mu}{3} - \frac{2\mu}{3}\right) = \lambda\theta.$$

Hence,

$$T_{11} - 2\mu e_{11} = \lambda\theta \text{ i.e., } T_{11} = \lambda\theta + 2\mu e_{11}$$

$$T_{22} - 2\mu e_{22} = \lambda\theta \text{ i.e., } T_{22} = \lambda\theta + 2\mu e_{22}$$

$$T_{33} - 2\mu e_{33} = \lambda\theta \text{ i.e., } T_{33} = \lambda\theta + 2\mu e_{33}$$

Above relations hold good for special choice of co-ordinate axes along the common principal directions of stress and strain. So, for any system of co-ordinate axes we have the transformation of stresses and strains

$$T'_{11} = T_{ij}l_il_j = T_{11}l_1^2 + T_{22}l_2^2 + T_{33}l_3^2$$

$$e'_{11} = e_{ij}l_il_j = e_{11}l_1^2 + e_{22}l_2^2 + e_{33}l_3^2$$

$$\therefore T'_{11} - 2\mu e'_{11} = T_{11}l_1^2 + T_{22}l_2^2 + T_{33}l_3^2 - 2\mu\left(e_{11}l_1^2 + e_{22}l_2^2 + e_{33}l_3^2\right)$$

$$= (T_{11} - 2\mu e_{11})l_1^2 + (T_{22} - 2\mu e_{22})l_2^2 + (T_{33} - 2\mu e_{33})l_3^2$$

$$= \lambda\theta\left(l_1^2 + l_2^2 + l_3^2\right) \text{ (using (i))}$$

$$= \lambda\theta. \quad \left(\because l_1^2 + l_2^2 + l_3^2 = 1\right)$$

Therefore,

$$T'_{11} = \lambda\theta + \lambda\mu e'_{11}$$

Similarly,

$$T'_{22} = \lambda\theta + 2\mu e'_{22}$$

$$T'_{33} = \lambda\theta + 2\mu e'_{33}$$

Hence the result.

Ex-3. *When a body is subjected to uniform pressure such that* $T_{11} = T_{22} = T_{33} = -p, T_{12} = T_{23} = T_{31} = 0$. *Prove that* $\frac{u_1}{x_1} = \frac{u_2}{x_2} = \frac{u_3}{x_3} = \frac{-p}{3K}$, *assuming that displacement and rotation at* $x_1 = 0, x_2 = 0, x_3 = 0$ *are zero.*

Ans. The strain-stress relation in terms of σ, E and Θ is

$$e_{ij} = \frac{1+\sigma}{E}T_{ij} - \frac{\sigma\Theta}{E}\delta_{ij}$$

where

$$\Theta = T_{11} + T_{22} + T_{33} = -p - p - p = -3p$$

$$\therefore e_{ij} = \frac{1+\sigma}{E} T_{ij} + \frac{3p\sigma}{E}\delta_{ij}$$

Now

$$e_{11} = \frac{1+\sigma}{E} T_{11} + \frac{3p\sigma}{E}\delta_{11}$$

$$= \frac{1+\sigma}{E}(-p) + \frac{3p\sigma}{E}$$

$$= -\left(\frac{1-2\sigma}{E}\right)p$$

$$= -\frac{p}{3K}\left(\because K = \frac{E}{3(1-2\sigma)}\right)$$

$$\therefore \frac{\partial u_1}{\partial x_1} = -\frac{p}{3K} = \text{constant.}$$

Int. $$u_1 = -\frac{p}{3K}x_1 + f_1(x_2, x_3) \text{........(i)}$$

Again,

$$e_{22} = \frac{1+\sigma}{E} T_{22} + \frac{3p\sigma}{E}\delta_{22} = -\frac{p}{3K}$$

and $$e_{33} = \frac{1+\sigma}{E} T_{33} + \frac{3p\sigma}{E}\delta_{33} = -\frac{p}{3K}$$

$$\Rightarrow \frac{\partial u_2}{\partial x_2} = \frac{\partial u_3}{\partial x_3} = -\frac{p}{3K}$$

By integration we get

$$u_2 = -\frac{p}{3K}x_2 + f_2(x_1, x_3) \text{........(ii)}$$

and $$u_3 = -\frac{p}{3K}x_3 + f_3(x_1, x_2) \text{.......(iii)}$$

As $T_{12} = T_{23} = T_{23} = 0.$

$$\therefore e_{12} = e_{23} = e_{23} = 0.$$

i.e. $\frac{\partial u_1}{\partial x_2}+\frac{\partial u_3}{\partial x_2}=0$

$\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}=0$

and $\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}=0$

or, $\frac{\partial f_1}{\partial x_2}+\frac{\partial f_2}{\partial x_1}=0, \frac{\partial f_2}{\partial x_3}+\frac{\partial f_3}{\partial x_2}=0, \frac{\partial f_3}{\partial x_1}+\frac{\partial f_1}{\partial x_3}=0$ .... (iv)

Since $u_1=0, u_2=0, u_3=0$ when $x_1=0, x_2=0, x_3=0$, therefore from (i), (ii), (iii) and (iv) we must have.

$f_1=0, f_2=0, f_3=0.$

Hence

$$u_1=-\frac{p}{3K}x_1, u_2=-\frac{p}{3K}x_2, u_3=-\frac{p}{3K}x_3.$$

i.e. $\frac{u_1}{x_1}=\frac{u_1}{x_2}=\frac{u_3}{x_3}=-\frac{p}{3K}.$

Ex.-4. *Show that the stress potential or strain-energy function is a homogeneous function of second degree.*

Ans. The stress potential or strain-energy function W is given by

$$T_i=\frac{\partial W}{\partial e_i}(i=1,2,...6) \text{ .... (i)}$$

where $T_1=T_{11}, T_2=T_{22}, T_3=T_{33}, T_4=T_{23}, T_5=T_{31}, T_6=T_{12}$

and $e_1=e_{11}, e_2=e_{22}, e_3=e_{33}, e_4=2e_{23}, e_5=2e_{31}, e_6=2e_{12}$.

Since W represents potential energy per unit volume stored up in the body by strain deformation alone, W must be a function of components of strain so that

$$W=W(e_1,e_2,....,e_6) \text{ .... (ii)}$$

Expanding W in a power series about (0,0,....,0). Then we get

$$W=W_0+C_ie_i+\frac{1}{2}C_{ij}e_ie_j+... \text{ .... (iii)}$$

where $W_0 = W(0,0,\ldots,0), C_i = \left(\frac{\partial W}{\partial e_i}\right)_{e_i=0}, C_{ij} = \left(\frac{\partial^2 W}{\partial e_i \partial e_j}\right)_{e_i=0}$ etc.

Since we are interested in the derivatives of W, so we can set $W_0 = 0$, and hence

$$W = C_i e_i + \frac{1}{2} C_{ij} e_i e_j \text{........ (iv)}$$

Using (iv) in (i), then we get

$$T_i = C_i + \frac{1}{2}\frac{\partial}{\partial e_i}\left(C_{pq} e_p e_q\right) + \ldots$$

$$= C_i + \frac{1}{2} C_{pq}\left(e_p \frac{\partial e_q}{\partial e_i} + e_q \frac{\partial e_p}{\partial e_i}\right) + \ldots$$

$$= C_i + \frac{1}{2} C_{pq}\left(e_p \delta_{qi} + e_q \delta_{pi}\right) + \ldots$$

$$= C_i + \frac{1}{2} e_p C_{pi} + \frac{1}{2} e_q C_{iq} + \ldots$$

$$= C_i + \frac{1}{2}\left(C_{ji} + C_{ij}\right) e_j + \ldots\ldots\ldots\ldots \text{(v)}$$

But for the continuum body, we assume that

$$\frac{\partial^2 W}{\partial e_i \partial e_j} = \frac{\partial^2 W}{\partial e_j \partial e_i}$$

$$\therefore C_{ij} = C_{ji}$$

Hence (v) becomes

$$T_i = C_i + C_{ij} e_j + \ldots$$

Also for elastic body, $T_i = 0$ whenever $e_j = 0$.

$$\therefore C_i = 0$$

Thus for linear elastic solid, stresses are linear functions of strain, we neglect all terms of order two and higher in strain.

So, $T_i = C_{ij} e_j$ .... (vi)

and then from (iv) we get

$$W = \frac{1}{2} C_{ij} e_i e_j \text{ .... (vii)}$$

where $C_{ij} = C_{ji}$.

Combining (vi) and (vii),

$$W = \frac{1}{2} T_i e_i \, (i = 1, 2, \ldots 6) \text{ .... (viii)}$$

$$= \frac{1}{2} T_{ij} \frac{de_{ij}}{de_i} (i, j = 1, 2, 3)$$

$$= \frac{1}{2} T_{ij} \rho_{ij} \, (i, j = 1, 2, 3) \text{ where } \rho_{ij} = \frac{de_{ij}}{de_j}$$

which is known as Clapeyron's formula.

Again from (viii), we get

$$2W = T_i e_i = \frac{\partial W}{\partial e_i} e_i \text{ (using (i))}$$

i.e. $$e_1 \frac{\partial W}{\partial e_1} + e_2 \frac{\partial W}{\partial e_2} + \ldots + e_6 \frac{\partial W}{\partial e_6} = 2W$$

which shows that W is a homogeneous function of second degree in $e_1, e_2, \ldots, e_6$.

Ex.5 *Show that Navier's equation of motion in absence of body forces* $(\lambda + \mu)\theta,_j + \mu \nabla^2 u_i = \rho \ddot{u}_i$ *is satisfied by* $\vec{u} = \vec{\nabla}\phi + \vec{\nabla} \times \vec{\psi}$ *provided* ϕ *and* $\vec{\psi}$ *satisfies three-dimension wave equation.*

$$\frac{\partial^2 \varphi}{\partial t^2} = c_1^2 \nabla^2 \varphi, \quad \frac{\partial^2 \vec{\psi}}{\partial t^2} = c_2^2 \nabla^2 \vec{\psi}.$$

Ans. According to the given condition

$$\vec{u} = \vec{\nabla}\varphi + \nabla^2 \times \vec{\psi}$$

we have

$$u_i = \phi,_i + e_{ijk} \psi_{k,j}$$

where $e_{ijk} = \begin{cases} 0, \text{ if any two of } i, j, k \text{ are equal} \\ 1, \text{ if } i, j, k \text{ are even permutation of } 1, 2, 3 \\ -1, \text{ if } i, j, k \text{ are odd permutation of } 1, 2, 3. \end{cases}$

Then Navier's equation of motion in absence of body forces reduces to

$$\mu(\varphi,_i + e_{ijk}\psi_{k,j}),_{ll} + (\lambda+\mu)(u_{l,l}),_i = \rho\ddot{u}_i$$

or, $\mu(\varphi,_{ill} + e_{ijk}\psi_{k,jll}) + (\lambda+\mu)(\varphi,_l + e_{ljk}\psi_{k,j}),_{li} = \rho(\ddot{\varphi},_i + e_{ijk}\ddot{\psi}_{k,j})$

But $e_{ijk}\psi_{k,jli} = e_{jlk}\psi_{k,ijl} = -e_{ljk}\psi_{k,ijl} = -e_{ijk}\psi_{k,jli}$.

Hence, $e_{ijk}\psi_{k,jli} = 0$.

Therefore above equation becomes

$$[(\lambda+\mu)\varphi,_{ill} - \rho\ddot{\varphi},_i] + e_{ijk}[\mu\psi_{k,jll} - \rho\ddot{\psi}_{k,j}] + \mu\varphi,_{ill} = 0$$

or, $[(\lambda+2\mu)\varphi,_{ll} - \rho\ddot{\varphi}]_{,i} + e_{ijk}[\mu\psi_{k,ll} - \rho\ddot{\psi}_k]_{,j} + \mu\phi,_{ill} = 0$

which is possible only when

$$(\lambda+2\mu)\varphi,_{ll} - \rho\ddot{\varphi} = 0$$

and $\mu\psi_{k,ll} - \rho\ddot{\psi}_k = 0$

i.e., $(\lambda+2\mu)\nabla^2\varphi = \rho\ddot{\varphi}$

and $\mu\nabla^2\psi_k = \rho\ddot{\psi}_k$

or, $\dfrac{\partial^2\varphi}{\partial t^2} = c_1^2\nabla^2\varphi$

and $\dfrac{\partial^2\psi_k}{\partial t^2} = c_2^2\nabla^2\psi_k$

where $c_1 = \sqrt{\dfrac{\lambda+2\mu}{\rho}}$ and $c_2 = \sqrt{\dfrac{\mu}{\rho}}$.

Ex.-6. *Show that, in case of equilibrium of elastic body under no body forces, the strain invariant $\theta = e_{ii}$ and stress invariant $\Theta = T_{ii}$ and rotation tensor $R_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$ are harmonic function whereas e_{ij}, T_{ij} and u_i are biharmonic functions.*

Ans. The strain components are given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

and stress-strain relation is

$$T_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij} \text{ where } \theta = e_{11} + e_{22} + e_{33} = e_{ii} = u_{k,k}$$

$$= \lambda u_{k,k}\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right)$$

$$\therefore T_{ij,j} = \lambda\left(u_{k,k}\right)_{,j}\delta_{ij} + \mu\left(u_{i,jj} + u_{j,ij}\right)$$

$$= \lambda\left(u_{k,k}\right)_{,i} + \mu\nabla^2 u_i + \mu\left(u_{j,j}\right)_{,i}$$

$$= \lambda\theta_{,i} + \mu\nabla^2 u_i + \mu\theta_{,i}$$

$$= (\lambda+\mu)\theta_{,i} + \mu\nabla^2 u_i$$

The equations of equilibrium of stresses are

$$T_{ij,j} + \rho F_i = 0$$

So, we have

$$(\lambda+\mu)\theta_{,i} + \mu\nabla^2 u_i + \rho F_i = 0$$

Hence equations of equilibrium in absence of body forces are

$$\mu\nabla^2 u_i + (\lambda+\mu)\theta_{,i} = 0 \text{ .... (i)}$$

or, $\mu\left(\nabla^2 u_i\right)_{,i} + (\lambda+\mu)\theta_{,ii} = 0$

or, $\mu\nabla^2\left(u_{i,i}\right) + (\lambda+\mu)\nabla^2\theta = 0 \quad \left[\because \frac{\partial}{\partial x_i}\left(\frac{\partial^2 u_i}{\partial x_i^2}\right) = \frac{\partial^2}{\partial x_i^2}\left(\frac{\partial u_i}{\partial x_i}\right)\right]$

or, $\mu\nabla^2\theta + (\lambda+\mu)\nabla^2\theta = 0$

or, $(\lambda+2\mu)\nabla^2\theta = 0$

$$\therefore \nabla^2\theta = 0. \ (\because \lambda + 2\mu \neq 0) \text{ .... (ii)}$$

Again from stress-strain relations we have

$$T_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}$$

$$\therefore T_{ii} = 3\lambda\theta + 2\mu e_{ii} = (3\lambda+2\mu)\theta$$

i.e., $\Theta = (3\lambda+2\mu)\theta$

So, $\nabla^2\Theta = (3\lambda+2\mu)\nabla^2\theta = 0$ (using (ii)) ...... (iii)

Again taking ∇^2 operation on (i), then we get

$$\mu\nabla^4 u_i + (\lambda+\mu)\nabla^2(\theta,i) = 0$$

or, $\nabla^4 u_i = -\left(\frac{\lambda+\mu}{\mu}\right)\nabla^2(\theta,_i) = -\left(\frac{\lambda+\mu}{\mu}\right)(\nabla^2\theta),_i$

$\therefore \nabla^4 u_i = 0$ [using (ii)] .... (iv)

Now Beltrami-Michell stress compatibility equations in absence of body forces are

$$\nabla^2 T_{ij} + \frac{1}{1+\sigma}\Theta,_{ij} = 0 \text{ [using equation (42)]}$$

or, $\nabla^4 T_{ij} + \frac{1}{1+\sigma}\nabla^2(\Theta,_{ij}) = 0$

or, $\nabla^4 T_{ij} + \frac{1}{1+\sigma}(\nabla^2\Theta),_{ij} = 0$

$\therefore \nabla^4 T_{ij} = 0$ (using (iii)) .... (iv)

Also, stress-strain relation can be written as

$$e_{ij} = \frac{(1+\sigma)}{E}\left[T_{ij} - \frac{\sigma}{1+\sigma}\Theta\delta_{ij}\right]$$

$$\therefore \nabla^4 e_{ij} = \frac{(1+\sigma)}{E}\left[\nabla^4 T_{ij} - \frac{\sigma}{1+\sigma}\delta_{ij}\nabla^4\Theta\right]$$

i.e., $\therefore \nabla^4 e_{ij} = 0$ (using (iii) and (iv)) ....(vi)

Again from (i), $\mu\nabla^2 u_{i,j} + (\lambda+\mu)\theta,_{ij} = 0$

$\therefore \mu\nabla^2 u_{j,i} + (\lambda+\mu)\theta,_{ji} = 0$ (interchanging i and j)

Subtracting above results, we get

$$\mu\nabla^2(u_{i,j} - u_{j,i}) = 0\left(\because \theta,_{ij} = \frac{\partial^2\theta}{\partial x_i \partial x_j} = \frac{\partial^2\theta}{\partial x_j \partial x_i} = \theta,_{ji}\right)$$

$\therefore \mu\nabla^2 R_{ij} = 0$ where $R_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$

i.e., $\nabla^2 R_{ij} = 0$ .... (vii)

Results (ii), (iii) and (vii) implies that θ, Θ and R_{ij} are harmonic functions, and results (iv) (v) and (vi) implies that e_{ij}, T_{ij} and u_i are biharmonic functions.

Ex. 7 *Show that the following stress components are not solutions of a problem in elasticity, even though they satisfy the equations of equilibrium with zero body forces* $T_{11}=\alpha\left[x_2^2+\sigma\left(x_1^2-x_2^2\right)\right]$, $T_{22}=\alpha\left[x_1^2+\sigma\left(x_2^2-x_1^2\right)\right]$, $T_{33}=\alpha\sigma\left(x_1^2+x_2^2\right)$, $T_{12}=-2\alpha\sigma x_1x_2, T_{23}=T_{31}=0.$

Ans. We know that the given stress components are the solutions of a problem in elasticity if they satisfy the equations of equilibrium

$$\nabla^2 T_{ij}+\frac{1}{1+\sigma}\Theta_{,ij}=-\frac{\sigma}{1+\sigma}\delta_{ij}\rho\, div.\vec{F}-\rho\left[F_{i,j}+F_{j,i}\right]$$

In absence of body forces it reduces into

$$\nabla^2 T_{ij}+\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x_i\partial x_j}=0.$$

Now $\nabla^2 T_{11}+\dfrac{1}{1+\sigma}\dfrac{\partial^2\Theta}{\partial x_1^2}=\left(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}+\dfrac{\partial^2}{\partial x_3^2}\right)T_{11}+\dfrac{1}{1+\sigma}\dfrac{\partial^2}{\partial x_1^2}(T_{11}+T_{22}+T_{33})$

$$=\left(2\alpha\sigma+(1-\sigma)2\alpha+0\right)+\frac{1}{1+\sigma}(2\alpha\sigma+2\alpha-2\alpha\sigma+2\alpha\sigma)$$

$$=2\alpha\frac{(1+\sigma)}{(1+\sigma)}+2\alpha=4\alpha$$

$\neq 0$ ($\because \alpha\neq 0$, otherwise all $T_{ij}=0$)

$\Rightarrow T_{11}$ is not a solution of a problem in elasticity.

Similarly, $\nabla^2 T_{22}+\dfrac{1}{1+\sigma}\dfrac{\partial^2\Theta}{\partial x_2^2}=\left(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}+\dfrac{\partial^2}{\partial x_3^2}\right)T_{22}+\dfrac{1}{1+\sigma}\dfrac{\partial^2}{\partial x_2^2}(T_{11}+T_{22}+T_{33})$

$$=\left[2\alpha(1-\sigma)+2\alpha\sigma\right]+\frac{1}{1+\sigma}[(1-\sigma)2\alpha+2\alpha\sigma+2\alpha\sigma]$$

$$=2\alpha+\frac{(1+\sigma)}{(1+\sigma)}\cdot 2\alpha=4\alpha\neq 0,$$

and we can show that

$$\nabla^2 T_{ij}+\frac{1}{1+\sigma}\Theta_{,ij}\neq 0 \text{ for } T_{33},T_{12},T_{23},T_{31}.$$

Hence given stress components are not solutions of the equations of equilibrium.

4.13 Self Assessment Questions

1. When a bar is stretched by a tension T such that $T_{11} = T, T_{22} = T_{33} = T_{23} = T_{31} = T_{12} = 0$; prove that

$$\frac{u_1}{x_1} = \frac{(\lambda+\mu)T}{\mu(3\lambda+2\mu)}, \frac{u_2}{x_2} = \frac{u_3}{x_3} = \frac{\lambda T}{2\mu(3\lambda+2\mu)}$$

assuming that displacement and rotation as $x_1 = 0, x_2 = 0, x_3 = 0$ are zero.

2. If $W = \frac{1}{2}\left[\lambda e_{kk}^2 + 2\mu e_{ij} e_{ij}\right]$ then prove the following:

i) $T_{ij} = \frac{\partial W}{\partial e_{ij}}$,

ii) $W = \frac{1}{2} T_{ij} e_{ij}$,

iii) W is a scalar invariant,

iv) $W \geq 0$ and $W = 0$ *iff* $e_{ij} = 0$,

v) $\frac{\partial W}{\partial T_{ij}} = e_{ij}$.

3. In an elastic beam placed along the x_3-axis and bent by a couple about the x_2-axis, the stresses are found to be $T_{33} = -\frac{E}{R} x_1, T_{11} = T_{22} = T_{12} = T_{23} = T_{31} = 0$, where R is a constant. Find the corresponding strains.

4. A beam placed along the x_3-axis and subjected to a longitudinal stress T_{11} at every point is so constrained that e_{22} and T_{33} are zero at every point. Show that

$$T_{22} = \gamma T_{11}, e_{11} = \frac{1-\gamma^2}{E} T_{11}, e_{33} = -\frac{\gamma(1+\gamma)}{E} T_{11}$$

5. In a vertical elastic beam deforming under its own weight (acting in the x_3-direction) the strain components are found to be

$$e_{11} = e_{22} = \frac{-\lambda}{2(\lambda+\mu)} \cdot a(b-x_3), e_{33} = a(b-x_3), e_{12} = e_{23} = e_{31} = 0,$$

where a, b are constants. Find the stress components.

6. Show that the stress-strain relation is equivalent to the following relations taken together:

$$tr(T_{ij}) = (3\lambda+2\mu) tr(e_{ij})$$

and $T_i^{(d)} = 2\mu e_i^{(d)}$

where $T_1^{(d)} = \frac{1}{3}(2T_1 - T_2 - T_3), T_2^{(d)} = \frac{1}{3}(2T_2 - T_3 - T_1)$.... etc.

and $e_1^{(d)} = \frac{1}{3}(2e_1 - e_2 - e_3), e_2^{(d)} = \frac{1}{3}(2e_2 - e_3 - e_1)$... etc.

are known as stress and strain deviators.

4.14 Further Suggested Reading :

1. Continuum Mechanics : T.J. Chung, Prentic-Hall.
2. Schaum's Outline of Theory and Problem of Continuum Mechanics : Gedrge R.Mase, McGraw-Hill.
3. Continuum Mechanics : A.J.M. Spencer, Longman.
4. Mathematical Theory of Continuum Mechanics : R.N. Chatterjee, Narosa Publishing House.
5. Foundation of Fluid Mechanics : S.W. Yuan, Prentice-Hall.
6. Textbook of Fluid Dynamics : F.Chorlton, CBS Publishers and Distributors.
7. Fluid Dynamics : J.K. Goyal, K.P. Gupta, Pragati Prakasan.
8. Theory of Elasticity : Yu. Amenzade, Mir Publishers, Moscow.
9. Applied Elasticity : C.T. Wang, McGraw-Hill.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I
Paper – V : Group – A : Marks – 50

Module No. - 53
(MECHANICS OF CONTINUOUS MEDIA)

STRUCTURE :

5.1 Introduction :

Perfect Fluid :

Before going to discuss about perfect fluid at first we shall discuss about the concept and behaviour of the fluid.

When a force of certain magnitude applied to a material produces a finite, small deformation, then the material is called a solid and if the deformation altering the shape of the material disappears as soon as force is removed so that material returns to its original shape, then the material is called *elastic solid.*

On the other hand when a large external force applied to the material produces a permanent deofrmation which does not completely vanish even after the removal of forces so that material does not fully recover its original shape, then the material is called *plastic* one.

But when the smallest external force applied to the material will eventually cause a continuous shear deformation of relative sliding so that its constituent particles are freely mobile, then the material is called *fluid material*. This continuous shear deformation of the fluid under the action of smallest force is called the *flow of the fluid.* So, *the fluid we mean a substance which is capable of flowing*. Actually fluids are divided into two categories : (i) liquids, (ii) gases. Actual fluids have five physical properties ; density, volume, temperature, pressure and viscosity. Unlike sollid, fluid has no stress-free state to which it eventually returns if the external force is removed. Consequently, every configuration can be regarded as the reference configuration. The continuous flow of fluids changes the shape of the fluid. It has no definte shape, but for given sufficient time it takes the shape of the container into which it is placed.

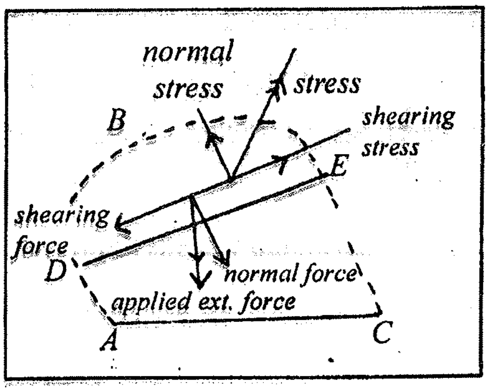
5.2 Objectives :

In this module, the students will learn baout perfect fluid, its velocity, acceleration, stream lines, path lines, type of motion, velocity potential, equation of motion, equation of continuity. Integrals of equation of motion etc.

5.3 Key words :

Fluid, Perfect fluid, Newtonian fluid, non-Newtonian fluid, Pseudo plastic, Lagrangian Method, Eulerian method, Local rate, Individual time rate, velocity, acceleration, Stream lines, Path lines, steady motion, rotational motion, irrotational motion, velocity potential, equation of continuity, boundary surface, constitutive equation, hydrostatic pressure, equations of motion, Conservative field, Pressure equation, integrals of Lagrange's equation of motion, Chauchy's integral, Helmhotlz Vorticity equation.

5.4 Behaviour of the Fluid :



(i) At rest : Let us consider a material ABC resting in equilibrium on a horizontal plane AC under the action of gravity. Let DE be an imaginary oblique plane section within the material ABC. The external force acting on the portion DBE is its weight acting vertically downwards and it can be resolved into two components: normal force perpendicular to the surface DE and shearing force along the surface DE. To balance these forces, portion $ADEC$ must exert stress on portion DBE across the surface DE. Again this stress can be resolved into two components: the normal stress perpendicular to the surface DE and shearing stress along DE.

If the material be solid then the solid, at rest, exerts shearing stress resting the sliding movement of one layer over another and thereby balance the applied shearing force to maintain the equilibrium of the portion DBE.

But if the material be fluid, then the portion DBE can not stay heaped up in the state of equilibrium position like a solid and consequently it will slide down the plane DE because intermolecular forces are weaker in fluid and fluid molecules can move past one another more freely than in a solid. Thus fluid at rest cannot exert any shearing stress on any adjacent layer to resist its sliding movement and therefore it can not support any shearing force, however small, so that static equilibrium between applied shearing force and shearing stress never works.

ii) In Motion : The fluid in motion exert on any adjacent layer moving with different velocity some kind of frictional resistance to alterations of form in the form of shearing stress in the tangential plane in addition to the normal stress in order to accelerate or dissipate its state of relative motion. So, when the fluid is at rest these shearing stresses have no role to play but only when some portion of it is moving on, they are required to oppose or accelerate the motion. The relative motion of the fluid layers gives rise to the shearing stresses, and these stresses vanish when the rate of deformation or the rate of change of strain is zero. In case of perfectly elastic solid, when a force is applied, it undergoes a finite deformation which vanishes as soon as force is removed.

This property of the fluid in motion is able to exert shearing stress on the adjacent layer to resist its sliding motion under the action of smallest shearing force is called *viscosity* and fluid having this property of viscosity is called *viscous fluid*.

Now we define some viscous fluids as:

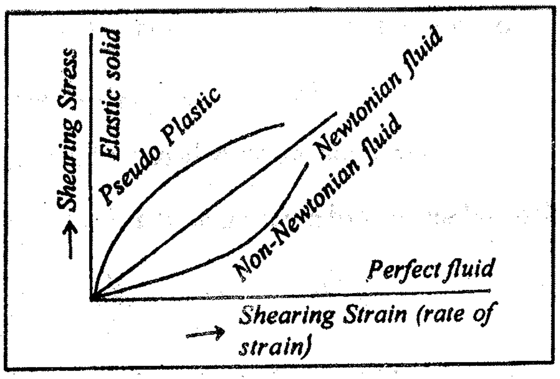
5.5 Viscous Fluid : A viscous fluid is a continuous material which can not develop any shearing stress when in equilibrium yet when in motion, it can exert appropriate shearing stress on any adjacent layer with which it is in contact to resist its sliding movement under the action of smallest shear force but in the long run, this smallest shear force causes a continuous shear deformation altering the shape of the material.

5.6 Newtonian Viscous Fluid : If the stress exerted by a viscous fluid is a linear function of the rate of strain, such that stress automatically vanishes when rate of strain is zero, then the fluid is called Newtonian viscous fluid or linearly viscous fluid.

5.7 Non-Newtonian Viscous Fluid : If the shearing stress is non-linear function of rate of strain, such that shearing stress is zero when rate of strain is zero, the fluid is called non-Newtonian viscous fluid.

5.8 Perfect Fluid : A perfect fluid is a continuous material that is incapable of exerting any shearing stress on any adjacent layer in its contact to resist its shearing movement under the action of the smallest shearing force, whether it is at rest or in motion so that continuous shear deformation readily takes place altering the shape of the material. The following have the same meaning: perfect, frictionless, inviscous, non-viscous and ideal. From the definition of shearing stress and body force it is clear that body force per unit area at every point of surface of a perfect fluid acts along the normal to the surface at that point.

Behaviour of a perfect fluid for which shearing stress is zero is represented by abscissa. Non-Newtonian fluid is represented by curve and main class of non-Newtonian fluids are Binghm plastic, Pseudo plastic and Dilants. Also the ordinate represents the perfectly elastic material because its deformation is finite and constant so that rate of strain is zero. But stress in a linearly elastic solid is proportional to its strain whereas stress is linearly viscous fluid is proportional to the time rate of strain.



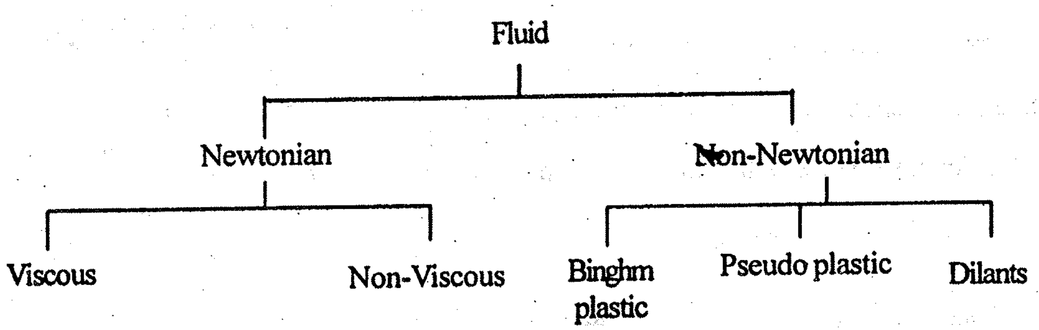
5.9 Classification of Fluids :

Fluids may be classified as : Compressible and incompressible w.r.t. the change of volume.

If a change in the stress applied to a quantity of fluid produce a change in volume so that fluid can easily be compressed then the fluid is said to be compressible (As the molecules of such fluid are far apart having considerably empty space betwen them). Gas is compressible fluid.

If a change in the stress produce no change in the volume of a fluid, so that it can not be compressed by any amount of stress, then the fluid is said to be incompressible (As the molecules of such fluid are close together with little space between them). Liquid is incompressible.

According to the behaviour of the fluid it can be classified as:



5.10 Methods of Descriptions of Motion :

i) Lagrangian method : In this method, any particle of the fluid is selected and its motion is studied and hence we determine the history of every fluid particle.

If (a, b, c) be the initial position of a fluid particle and after some time t let takes the position (x, y, z). Then it is obvious that x, y, z are function of t, a, b, c. Thus

$$x = f_1(a,b,c,t), y = f_2(a,b,c,t), z = f_3(a,b,c,t).$$

If the motion is everywhere continuous, then f_1, f_2, f_3 are continuous functions so that we can assume that first and second order partial derivatives w.r.t a, b, c, t exist. Then velocity and acceleration components are

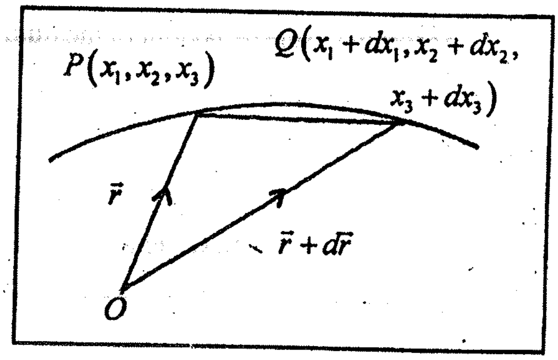
$$\dot{x} = \frac{\partial f_1}{\partial t}, \dot{y} = \frac{\partial f_2}{\partial t}, \dot{z} = \frac{\partial f_3}{\partial t}; \ddot{x} = \frac{\partial^2 f_1}{\partial t^2}, \ddot{y} = \frac{\partial^2 f_2}{\partial t^2}, \ddot{z} = \frac{\partial^2 f_3}{\partial t^2}.$$

ii) Eulerian method : In this method, any point fixed in the space occupied by a fluid is selected and we observe the change in the state of the fluid as the fluid passes through this point. Since the point is fixed and so x, y, z, t are independent variables. Hence the symbols $\dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z}$ have no meaning.

Note : Difference between Eulerian and Lagrangian methods are :

In Eulerian method we study the motion of every fluid particle at a fixed point; whereas in Lagrangian method we study the motion of every fluid particle at every point. Hence Euler's method corresponds to local time rate of change and Lagrangian method corresponds to invidual time rate of change.

5.11 Local and Particle Rate (Individual time rate) of Change :



Suppose a particle of fluid moves from $P(x_1, x_2, x_3)$ at time t to $Q(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ at time $t + \delta t$. Let $A(x_1, x_2, x_3, t)$ be a scalar function associated with some property of the fluid (e.g., the density). Then in the motion of the particle from P to Q, the total change of A is given by

$$\delta A = \frac{\partial A}{\partial x_1}\delta x_1 + \frac{\partial A}{\partial x_2}\delta x_2 + \frac{\partial A}{\partial x_3}\delta x_3 + \frac{\partial A}{\partial t}\delta t$$

Thus the total rate of change of A at the point P at time t in the motion of the particle is

$$\frac{dA}{dt} = \lim_{\delta t \to 0}\left(\frac{\delta A}{\delta t}\right) = \frac{\partial A}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial A}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial A}{\partial x_3}\cdot\frac{dx_3}{dt} + \frac{\partial A}{\partial t}$$

$$= v_1 \frac{\partial A}{\partial x_1} + v_2 \frac{\partial A}{\partial x_2} + v_3 \frac{\partial A}{\partial x_3} + \frac{\partial A}{\partial t}$$

where $\vec{V} = (v_1, v_2, v_3)$ is the velocity of the fluid particle at P with

$v_i = \frac{dx_i}{dt} (i = 1,2,3)$. Hence

$$\frac{dA}{dt} = \vec{V} \cdot \vec{\nabla} A + \frac{\partial A}{\partial t} \text{ .... (1)}$$

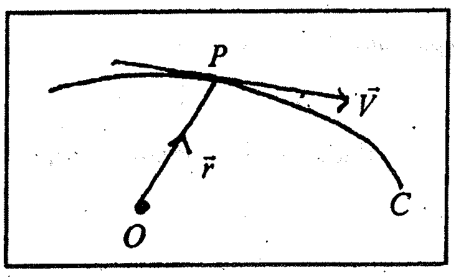
and $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla}$ .... (2)

Also (1) is valid for a vector function $\vec{F}(x_1, x_2, x_3, t)$ associated with some property of the fluid (e.g., velocity). Then

$$\frac{d\vec{F}}{dt} = \frac{\partial \vec{F}}{\partial t} + (\vec{V} \cdot \vec{\nabla}) \vec{F} \text{ .... (3)}$$

Here $\frac{dA}{dt}, \frac{d\vec{F}}{dt}$ are total time differentiations following the fluid particle and are called the particle rates of change or individual time rate. Also the partial time derivates $\frac{\partial A}{\partial t}, \frac{\partial \vec{F}}{\partial t}$ are only the time rates of change at $P(x_1, x_2, x_3)$ considered fixed in space : they are the local rates of change. Now $\vec{V} \cdot \vec{\nabla} A$ or $(\vec{V} \cdot \vec{\nabla})\vec{F}$ represents the rate of change due solely to the motion of the particle along its path.

5.12 Acceleration : Let the fluid particle be travelling along a curve C. At time t its position P is specified by $\overline{OP} = \vec{r}$ and its velocity $\vec{V}$ along the tangent at P to C is in the direction of the particle's motion. Then the instantaneous acceleration $\vec{f}$ at $P(x_1, x_2, x_3)$ is



$$\vec{f} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla}) \vec{V} \text{ .... (4)}$$

If $\vec{V} = (v_1, v_2, v_3)$ then we have the components of acceleration are

$$\frac{dv_1}{dt}=\frac{\partial v_1}{\partial t}+\left(v_1\frac{\partial}{\partial x_1}+v_2\frac{\partial}{\partial x_2}+v_3\frac{\partial}{\partial x_3}\right)v_1$$

$$\frac{dv_2}{dt}=\frac{\partial v_2}{\partial t}+\left(v_1\frac{\partial}{\partial x_1}+v_2\frac{\partial}{\partial x_2}+v_3\frac{\partial}{\partial x_3}\right)v_2$$

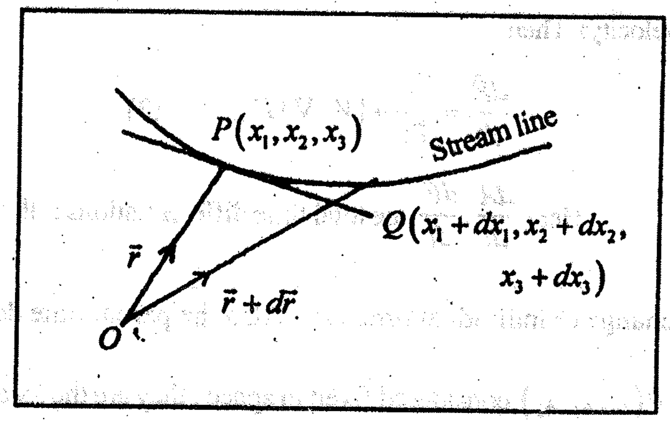
$$\frac{dv_3}{dt}=\frac{\partial v_3}{\partial t}+\left(v_1\frac{\partial}{\partial x_1}+v_2\frac{\partial}{\partial x_2}+v_3\frac{\partial}{\partial x_3}\right)v_3$$

In tensor form, with co-ordinates x_i and velocity components $v_i (i=1,2,3)$ then (5) can be written as

$$\frac{dv_i}{dt}=\frac{\partial v_i}{\partial t}+v_j v_{i,j} \text{ ....(6)}$$

5.13 Stream lines and Path lines :

A stream line or line of flow is a curve such that the tangent at any point of it at any instant of time coincides with the direction of the motion of the fluid at that point. If $\vec{V}=(v_1,v_2,v_3)$ be the velocity of the fluid particle (material point) at any point $P(x_1,x_2,x_3)$ on the stream line at time t and $\overline{OP}=\vec{r}$ then $\overline{PQ}$, where $Q(x_1+dx_1,x_2+dx_2,x_3+dx_3)$ a is neighbour-ing point of P on the stream line, will be the tangent to the stream line when $Q\to P$, i.e., direction of the tangent and direction of velocity are parallel, i.e., $\vec{V}$ is parallel to $d\vec{r}$ and hence



$$\vec{V}\times d\vec{r}=\vec{0}$$

If follows that the stream lines are the solutions of the differential equatins $\vec{V}=k\,d\vec{r}$, k being constant

$$\text{i.e., } \frac{dx_1}{v_1}=\frac{dx_2}{v_2}=\frac{dx_3}{v_3} \text{ ....(7)}$$

These equations yield a double infinity of solutions, at time t.

A path line is a curve which a particular fluid particle describes during its motion. The differential equations of path lines are

$$\frac{d\vec{r}}{dt} = \vec{V} \text{ where } \vec{r} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$$

which implies $\frac{dx_1}{dt} = v_1, \frac{dx_2}{dt} = v_2, \frac{dx_3}{dt} = v_3$

i.e. $\frac{dx_i}{dt} = v_i$ .... (8)

These equations have a triply-infinite set of solutions.

Stream lines show how each fluid particle is moving at a given instant whereas the path lines show how a given fluid particle is moving at each instant.

For steady flow, i.e., when $\frac{\partial \vec{V}}{\partial t} = \vec{0}$ then $\vec{V} = \vec{V}(x_1, x_2, x_3)$ which implies that stream lines do not vary with time t and coincide with path lines.

5.14 Types of Motion :

Steady Motion : A fluid motion is said to be steady if the condition at any point in the fluid at any time remains the same for all time, i.e., a fluid motion is said to be steady if

$$\frac{\partial \rho}{\partial t} = 0, \frac{\partial p}{\partial t} = 0, \frac{\partial \vec{V}}{\partial t} = 0,$$

where $\rho, p, \vec{V}$ denotes respectively density, pressure, velocity.

Rotational Motion : A fluid motion is said to be rotational if $\vec{W} = \vec{\nabla} \times \vec{V} \neq \vec{0}$ at every time and at every point.

Irrotational Motion : A fluid motion is said to be irrotational if $\vec{W} = \vec{\nabla} \times \vec{V} = \vec{0}$ at every point and at every time.

5.15 Velocity Potential : When the fluid velocity at time t is $\vec{V} = (v_1, v_2, v_3)$ in Cartesians, the equations of the stream lines at that instant are

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3}$$

These curves cut the surfaces

$$v_1 dx_1 + v_2 dx_2 + v_3 dx_3 = 0$$

orthogonally. Now suppose that at the considered instant t, we can find a scalar function $\varphi(x_1,x_2,x_3,t)$, uniform throughout the entire field of flow and such that

$$v_1dx_1+v_2dx_2+v_3dx_3=-d\varphi \;.......\;(9)$$

Then the expression on the L.H.S. of (9) is an exact differential and

$$v_1=-\frac{\partial\varphi}{\partial x_1},v_2=-\frac{\partial\varphi}{\partial x_2},v_3=-\frac{\partial\varphi}{\partial x_3},-\frac{\partial\varphi}{\partial t}=0.$$

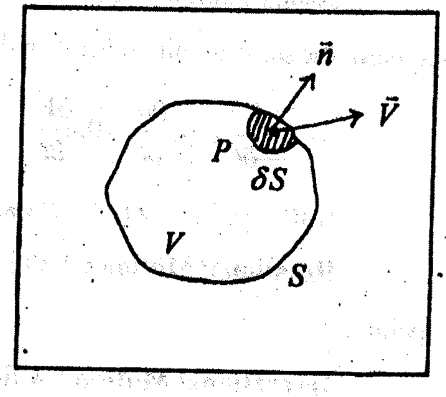
$$\therefore \vec{V}=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}=-\left[\hat{i}\frac{\partial}{\partial x_1}+\hat{j}\frac{\partial}{\partial x_2}+\hat{k}\frac{\partial}{\partial x_3}\right]\phi$$

$$=-\vec{\nabla}\phi \;..........\;(10)$$

This scalar function ϕ is termed as the *velocity potential.*

Here note that, $\varphi(x_1,x_2,x_3,t)=$ constant, represents the surfaces of equipotential.

5.16 Equation of Continuity : Let us consider a fixed closed surface S, enclosing a volume V in the region occupied by a moving fluid. This region of fluid contains neither sources nor sinks, i.e.,, there are no inlets or outlets through which fluid can enter of leave the region, the amount of fluid within the region is conserved in accordance with the principle of conservation of matter.



Let $\vec{n}$ be a unit outward normal vector drawn on the surface element δS, where fluid velocity is $\vec{V}$ and fluid density ρ.

Then the inward normal velocity is $(-\vec{n}\cdot\vec{V})$. Mass of the fluid entering across the surface S in unit time is

$$\int_S \rho(-\vec{n}\cdot\vec{V})dS=-\int_S \vec{n}\cdot(\rho\vec{V})dS=-\int_V \vec{\nabla}\cdot(\rho\vec{V})dV \;.........\;(11)$$

(by Gauss's div. theorem)

The mass of the fluid within the volume V is

$$M=\int_V \rho\, dV$$

Therefore, the rate of generation of the fluid within the volume is

$$\frac{\partial}{\partial t}M = \frac{\partial}{\partial t}\int_V \rho\, dV = \int_V \frac{\partial \rho}{\partial t} dV \text{ .... (12)}$$

$$\left[\because \rho\frac{\partial}{\partial t}(dV) = \rho d\left(\frac{\partial V}{\partial t}\right) = \rho \cdot 0 = 0 \text{ as volume is constant w.r.t. time}\right]$$

Now from the concept of conservation of mass which yields the equiation of continuity, we have

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \vec{\nabla}\cdot\left(\rho\vec{V}\right) dV$$

or, $$\int_V \left[\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right)\right] dV = 0$$

since V is an arbitrary, therefore $dV \neq 0$ and hence above integrand vanishes, i.e.,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0 \text{ ....(13)}$$

which is known as Eulerian equation of continuity.

Again from (13),

$$\frac{\partial \rho}{\partial t} + \vec{V}\cdot\vec{\nabla}\rho + \rho\vec{\nabla}\cdot\vec{V} = 0$$

or, $$\left(\frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla}\right)\rho + \rho\vec{\nabla}\cdot\vec{V} = 0$$

or, $$\frac{d\rho}{dt} + \rho\vec{\nabla}\cdot\vec{V} = 0 \text{ .... (14)}$$

or, $$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{V} = 0$$

or, $$\frac{d}{dt}\left(\log_e \rho\right) + \vec{\nabla}\cdot\vec{V} = 0 \text{ .... (15)}$$

Lagrange's Method :

Let (X_1, X_2, X_3) be the initial position of a fluid particle at time $t = t_0$ with volume dV_0 and density ρ_0. Let (x_1, x_2, x_3) be the position of the fluid particle after a time t with volume dV and density ρ. Since the mass of the fluid element remains unchanged during the motion, therefore,

$$\rho_0 dV_0 = \rho dV$$

or, $\rho_0 dX_1 dX_2 dX_3 = \rho dx_1 dx_2 dx_3$

or, $\rho_0 = \rho \dfrac{dx_1 dx_2 dx_3}{dX_1 dX_2 dX_3}$

$\therefore \rho J = \rho_0$ ....(16)

Where $J = \dfrac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)}$. (16) is the required equation of continuty.

Note 1. For incompressible fluid, we have $\dfrac{d\rho}{dt} = 0$. Then, from (14), we have

$$\rho \vec{\nabla} \cdot \vec{V} = 0$$

i.e., $\vec{\nabla} \cdot \vec{V} = 0$

Which is the equation of continuty in case of incompressible fluid.

Note 2. If $\vec{V}$ be the solenoidal vector, then $\vec{\nabla} \cdot \vec{V} = 0$

So, when the fluid is incompressible then the velocity vector is solenoidal vector.

Note 3. If the fluid motion be irrotational and fluid is incompressible, then we have

$\dfrac{d\rho}{dt} = 0$ (for incompressible fluid)

and $\vec{V} = -\vec{\nabla}\phi$ (for irrotational motion)

So, equation of continuity (14), becomes

$$0 + \rho \vec{\nabla} \cdot \left(-\vec{\nabla}\phi\right) = 0$$

or, $\nabla^2 \phi = 0$

which is the equation of continuity in this case, i.e., the equation of continuity reduces to Laplace's equation when the fluid is incompressible and irrotational.

Note 4. If the motion be symmetrical, then the velocity has only one component, say, v_1. Then the equation of continuity takes the following form :

$$\left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1}\right)\rho + \rho \frac{\partial v_1}{\partial x_1} = 0$$

where $\frac{d}{dt}=\frac{\partial}{\partial t}+v_1\frac{\partial}{\partial x_1}, \vec{V}=v_1\hat{i}, \vec{\nabla}=\hat{i}\frac{\partial}{\partial x_1}$.

Note 5. Equivalence between Eulerian and Lagrangian form :

From Lagragian equation of continuity, we have

$\rho J=\rho_0$ where $J=\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}$.

or, $\frac{d}{dt}(\rho J)=\frac{d}{dt}\rho_0=0$

or, $J\frac{d\rho}{dt}+\rho\frac{dJ}{dt}=0$

or, $J\frac{d\rho}{dt}+\rho J\vec{\nabla}\cdot\vec{V}=0 \quad \left(\because \frac{dJ}{dt}=J\vec{\nabla}\cdot\vec{V}\right)$

or, $\frac{d\rho}{dt}+\rho\vec{\nabla}\cdot\vec{V}=0$

which is Eulerian equation of continuity.

Again from Eulerian equation of continuity,

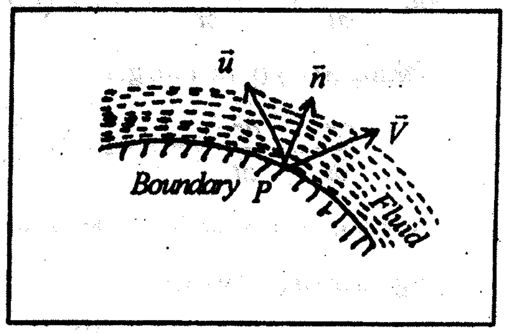
or, $\frac{d\rho}{dt}+\rho\vec{\nabla}\cdot\vec{V}=0 \quad \left(\because \frac{dJ}{dt}=J\vec{\nabla}\cdot\vec{V}\right)$

or, $J\frac{d\rho}{dt}+\rho\frac{dJ}{dt}=0$

or, $\frac{d}{dt}(\rho J)=0$

$\Rightarrow \rho J=$ Constant $=\rho_0$, say

which is Lagrangian equation of continuity.



5.17 Boundary Surface :

Let P be any arbitrary point on the boundary surface $F(\vec{r},t)=0$; let $\vec{V}$ and $\vec{u}$ the fluid velocity and velocity of the surace at P. Now the contact between the fluid and the surface will be maintained if the fluid and surface have the same velocity along the normal to the surface. So, the normal component of velocity of fluid is equal to normal component of the velocity of surface,

i.e., $\vec{V}\cdot\vec{n}=\vec{u}\cdot\vec{n}$ .... (17)

Again, $\vec{\nabla}F$ is normal to the surface $F(\vec{r},t)=0$. Hence $\vec{n}$ and $\vec{\nabla}F$ both are parallel to each other. As

$$\vec{n}=\frac{\vec{\nabla}F}{|\vec{\nabla}F|}$$

therefore from (17), we get

$$\vec{V}\cdot\frac{\vec{\nabla}F}{|\vec{\nabla}F|}=\vec{u}\cdot\frac{\vec{\nabla}F}{|\vec{\nabla}F|}$$

i.e., $\vec{V}\cdot\vec{\nabla}F=\vec{u}\cdot\vec{\nabla}F$ .... (18)

Let the point $P(\vec{r},t)$ moves to a point $Q(\vec{r}+\delta\vec{r},t+\delta t)$ in time δt. Since Q lies on the surface $F(\vec{r},t)=0$, therefore,

$$F(\vec{r}+\delta\vec{r},t+\delta t)=0$$

or, $F(\vec{r},t)+\left(\delta\vec{r}\cdot\vec{\nabla}F+\delta t\cdot\frac{\partial F}{\partial t}\right)=0$ (by Taylor's theorem)

$$\left[\because f(x+h,y+k)=f(x,y)+\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)+....\right]$$

or, $\frac{\delta\vec{r}}{\delta t}\cdot\vec{\nabla}F=\frac{\partial F}{\partial t}=0\quad(\because F(\vec{r},t)=0)$

Taking $\delta t\to 0$ then we get

$$\frac{\partial F}{\partial t}=-\frac{d\vec{r}}{dt}\cdot\vec{\nabla}F=-\vec{u}\cdot\vec{\nabla}F \text{ .... (19)}$$

where $\vec{u}$ is the velocity of the surface.

Again from (18) we get

$$\vec{V}\cdot\vec{\nabla}F=\vec{u}\cdot\vec{\nabla}F=-\frac{\partial F}{\partial t} \text{ (using (19)) .... (20)}$$

or, $\left(\frac{\partial}{\partial t}+\vec{V}\cdot\vec{\nabla}\right)F=0$

or, $\frac{dF}{dt}=0$ .... (21)

which is the required condition for the surface to be a possible form of boundary surface.

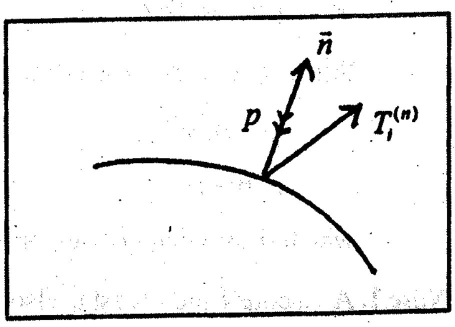
If the surface is rigid surface, then the condition becomes

$$\vec{V} \cdot \vec{\nabla} F = 0$$

Now the normal component of velocity for the boundary is

$$\vec{V} \cdot \vec{n} = \vec{V} \cdot \frac{\vec{\nabla} F}{|\vec{\nabla} F|} = \frac{-\frac{\partial F}{\partial t}}{|\vec{\nabla} F|} \text{ (using (20)).}$$

5.18 Constitutive Equation : Perfect fluid :



It is clear that a perfect fluid is characterized by the fact that it is incapable of exerting any shearing stress on the adjacent layers in its contact in resisting the shearing movement under a very small shearing force. Thus, the stress vector exerted by a perfect fluid must be normal to the surface. This normal stress which is always compressive is known as *pressure*.

If $T_i^{(n)}$ $(i = 1,2,3)$ are the components of the stress vector acting across a plane whose normal unit vector is $\vec{n}$, and p is the pressure that plane, then we have

$$T_i^{(n)} = -pn_i \ (i = 1,2,3)$$

Also the stress tensor T_{ij} are given by

$$T_i^{(n)} = T_{ij} n_j$$

$$\therefore T_{ij} n_j = T_i^{(n)} = -pn_i = -p\delta_{ij} n_j$$

or, $\quad T_{ij} = -p\delta_{ij}$ .... (23)

These equations are the constitutive equations of perfect fluid.

Tehorem : *The pressure at a point in a perfect fluid has the same magnitude in every direciton.*

Proof. Let P be any point in the perfect fluid. Let dS and dS' be any two arbitrary surface elements passing through P with normals n_i and n_i'. Let $T_i^{(n)}$ and $T_i^{(n')}$ be two stresses vectors acting at P across dS and dS'. Let T_{ij} be stress tensor at P. Let p and p' be the pressures representing compressive normal stresses at P across dS and dS'. Therefore

$$T_i^{(n)} = -pn_i \text{ and } T_i^{(n')} = -p'n'_i$$

i.e., $T_{ij}n_j = -pn_i$ and $T_{ij}n'_j = -p'n'_i$ .... (24)

Multiplying first and second equation of (24) by n'_i and n_i respectively, then we get

$$T_{ij}n_jn'_i = -pn_in'_i \text{ and } T_{ij}n'_jn_i = -p'n'_in_i$$

But $T_{ij}n'_jn_i = T_{ji}n'_in_j$ (interchanging dummy indices)

$$= T_{ij}n'_in_j \left(\because T_{ij} = T_{ji}\right)$$

$$-pn_in'_i = -p'n'_in_i$$

or, $\left(p-p'\right)n_in'_i = 0$.

Since n_i and n'_i are arbitrary, therefore

$$p-p' = 0$$

$$\therefore p = p'$$

which shows that pressure at a point of a perfect fluid has the same magnitude in every direction.

Note 1. A viscous fluid at rest is also incapable of exerting shear stress. If p_0 be the pressure at a point, same in every direction, then stress tensor T_{ij} will be given by

$$T_{ij} = -p_0\delta_{ij}$$

when fluid as at rest. This p_0 is called *hydrostatic pressure*.

Note 2. It should be noted that since the viscous fluid can exert shearing stress when it is in motion, the above theorem does not remain valid. In fact, the notion of fluid pressure as the compressive normal stress, i.e., in the sense of hydrostatic pressure may not hold good. Still one can define the pressure p at a point in a viscous fluid in motion as the average normal compressive stress at the point, i.e.,

$$p = -\frac{1}{3}\left(T_{11}+T_{22}+T_{33}\right) = -\frac{1}{3}T_{ii}$$

which is also known as *mechanical definition of pressure*.

5.19 Equations of Motion :

i) Euler's equation of Motion : From the principle of conservation of linear momentum, we have already deduced the Cauchy's first equation of motion of the continuum as,

$$T_{ij,j} + \rho F_i = \rho\frac{dv_i}{dt}$$

where T_{ij} is the stress tensor, F_i the body force per unit mass, v_i the velocity of continuum, and

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\vec{V} \cdot \vec{\nabla})$$

Also the constitutive equation for perfect fluid is

$$T_{ij} = -p\delta_{ij}$$

where p is the pressure at a point same in all direction. Hence from above two equations we get

$$(-p\delta_{ij})_{,j} + \rho F_i = \rho \frac{dv_i}{dt}$$

or, $$-p_{,j}\delta_{ij} + \rho F_i = \frac{dv_i}{dt}$$

or, $$-p_{,i} + \rho F_i = \frac{dv_i}{dt}$$

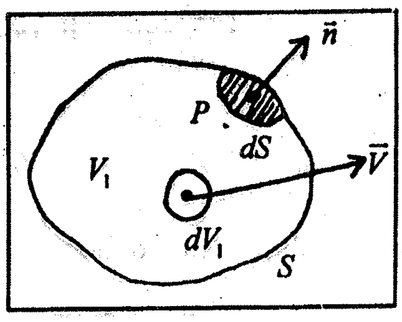
i.e., $$\frac{dv_i}{dt} = F_i - \frac{1}{\rho} p_{,i}$$

In vector notation,

$$\frac{d\overline{V}}{dt} = \overline{F} - \frac{1}{\rho}\overline{\nabla} p$$

or $$\frac{d\overline{V}}{dt} + (\overline{V}.\overline{\nabla})\overline{V} = \overline{F} - \frac{1}{\rho}\overline{\nabla} p \quad \text{...... (25)}$$

which is known as Euler's equation of motion for perfect fluid.



Alternately : Let a closed surface S enclosing a volume V_1 of a perfect fluid be moving with the fluid so that S contains the same number of fluid particle at any time t. Consider a point P inside S. Let ρ be the fluid density, $\overline{V}$ the fluid velocity and dV_1 the elementary volume enclosing P. Since the mass ρdV_1 remains unchanged throughout the motion so that $\frac{d}{dt}(\rho dV_1) = 0$

Hence the total momentum M_1 of the volume V_1 is

$$M_1 = \int_{V_1} \overline{V}(\rho dV_1)$$

$$\therefore \frac{dM_1}{dt} = \frac{d}{dt}\int_{V_1} \vec{V}\rho dV_1 = \int_{V_1}\left[\frac{d\vec{V}}{dt}\rho dV_1 + \frac{d}{dt}(\rho dV_1)\vec{V}\right]$$

$$= \int_{V_1}\frac{d\vec{V}}{dt}\rho dV_1$$

Let $\vec{n}$ be the unit outward normal vector on the surface element dS. Suppose $\vec{F}$ is the external force per unit mass acting on the fluid and p the pressure at any point on the element dS. Then total surface force is $\int_S p\left(-\vec{n}dS\right)$. Hence the total forces acting on the fluid is

$$\int_{V_1}\vec{F}(\rho dV_1) + \int_S p\left(-\vec{n}\right)dS$$

$$= \int_{V_1}\vec{F}\rho dV_1 + \int_S\left(-\vec{\nabla}p\right)dV_1 \qquad \text{(by Gauss's div.theorem)}$$

$$= \int_{V_1}\left(\rho\vec{F} - \vec{\nabla}p\right)dV_1$$

Now, by Newton's 2nd law of motion, we have,

rate of change of momentum = total applied force,

i.e., $\frac{d}{dt}M_1 = \int\left(\rho\vec{F} - \vec{\nabla}p\right)dV_1$

or, $\int_{V_1}\frac{d\vec{V}}{dt}\rho dV_1 = \int\left(\rho\vec{F} - \vec{\nabla}p\right)dV_1$

or, $\int_{V_1}\left[\rho\frac{d\vec{V}}{dt} - \rho\vec{F} + \vec{\nabla}p\right]dV_1 = 0$

Since S is an arbitrary and so V_1 is an also and hence $dV_1 \neq 0$ i.e.,

$$\rho\frac{d\vec{V}}{dt} - \rho\vec{F} + \vec{\nabla}p = 0$$

or, $\frac{d\vec{V}}{dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p$

Note : Since $\vec{\nabla}\left(\vec{V}\cdot\vec{V}\right) = 2\left[\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right) + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right]$

so, equation (25) can be expressed as

$$\frac{\partial\vec{V}}{\partial t}+\vec{\nabla}\left(\frac{1}{2}|\vec{V}|^2\right)-\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)=\vec{F}-\frac{1}{\rho}\vec{\nabla}p \quad \text{............ (26)}$$

or, $$\frac{\partial\vec{V}}{\partial t}+\vec{\nabla}\left(\frac{1}{2}|\vec{V}|^2\right)+\vec{W}\times\vec{V}=\vec{F}-\frac{1}{\rho}\vec{\nabla}p$$

where $\vec{W}=\vec{\nabla}\times\vec{V}$ and this equation of motion is known as *Lamb's hydrodynamical equation, the chief advantage of this equation is that it is invariant under a change of co-ordinate system.*

ii) **Lagrange's equation of motion :** Let (X_1,X_2,X_3) be the initial co-ordinates of a fluid particle at time t=0, and let this particle occupies the position (x_1,x_2,x_3) at time t. Then the acceleration components of this fluid particle at time t are $\frac{\partial^2x_1}{\partial t^2},\frac{\partial^2x_2}{\partial t^2},\frac{\partial^2x_3}{\partial t^2}$. Consequently equations of motion (25) reduces to

$$\frac{\partial^2x_i}{\partial t^2}=F_i-\frac{1}{\rho}\frac{\partial p}{\partial x_i}(i=1,2,3) \quad \text{........ (27)}$$

In Lagrangian method (X_1,X_2,X_3,t) are four independent variables so, we convert the partial derivatives w.r.t. $x_i(i=1,2,3)$ into those variables w.r.t $X_i(i=1,2,3)$ as follows:

$$\frac{\partial^2x_i}{\partial t^2}\cdot\frac{\partial x_i}{\partial X_K}=F_i\frac{\partial x_i}{\partial X_K}-\frac{1}{\rho}\frac{\partial p}{\partial x_i}\cdot\frac{\partial x_i}{\partial X_k}$$

(multiplying (27) by $\frac{\partial x_i}{\partial X_K}$ and sum over i)

or, $$\left(F_i-\frac{\partial^2x_i}{\partial t^2}\right)\frac{\partial x_i}{\partial X_k}=\frac{1}{\rho}\frac{\partial p}{\partial X_k}(k=1,2,3) \quad \text{.......... (28)}$$

More explicitly,

$$\left(F_i-\frac{\partial^2x_i}{\partial t^2}\right)\frac{\partial x_i}{\partial X_1}=\frac{1}{\rho}\frac{\partial p}{\partial X_1}$$

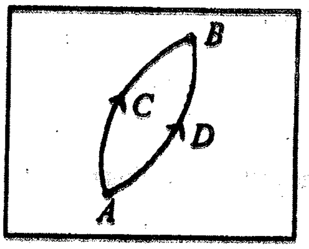
$$\left(F_i - \frac{\partial^2 x_i}{\partial t^2}\right)\frac{\partial x_i}{\partial X_2} = \frac{1}{\rho}\frac{\partial p}{\partial X_2},$$

$$\left(F_i - \frac{\partial^2 x_i}{\partial t^2}\right)\frac{\partial x_i}{\partial X_3} = \frac{1}{\rho}\frac{\partial p}{\partial X_3}$$

which are equations of motion of perfect fluid in Lagrangian method.

5.20 Convservative field of force : In a conservative field of force, the work done by a force $\vec{F}$ in taking a unit mass from a point A to a point B is independent of the path, i.e.,

$$\int_{ACB} \vec{F}\cdot d\vec{r} = \int_{ADB} \vec{F}\cdot d\vec{r} = -\Omega, \text{ (say)}$$



Here Ω is a scalar function and is known as *potential function.* And necessary and sufficient condition for the field $\vec{F}$ to be conservative if there exists a uniform differentiable scalar function Ω such that $\vec{F} = -\vec{\nabla}\Omega$.

5.21 Bernoulli's Equation of Motion (Pressure equation) :

When velocity potential exists and forces are conservative and derivable from a potential Ω, the equation of motion can always be integrated and the solution is $\int \frac{dp}{\rho} - \frac{\partial \varphi}{\partial t} + \frac{1}{2}V^2 + \Omega = F(t)$.

The Euler's equation of motion is

$$\frac{d\vec{V}}{dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p$$

and equation of continuity is

$$\frac{d\rho}{dt} + \rho\vec{\nabla}\cdot\vec{V} = 0$$

We have to make the assumption that pressure p is a function of density ρ alone so that $\int \frac{dp}{\rho}$ exists. Let us introduce pressure potential

$$P = \int_0^p \frac{dp}{\rho} \qquad \text{.........................} (29)$$

Then $\frac{dP}{dp} = \frac{1}{\rho}$

or, $\frac{\partial P}{\partial x_1}\frac{dx_1}{dp}+\frac{\partial P}{\partial x_2}\cdot\frac{dx_2}{dp}+\frac{\partial P}{\partial x_3}\cdot\frac{dx_3}{dp}+\frac{\partial P}{\partial t}\cdot\frac{dt}{dp}=\frac{1}{\rho}$

or, $\frac{\partial P}{\partial x_1}dx_1+\frac{\partial P}{\partial x_2}dx_2+\frac{\partial P}{\partial x_3}dx_3+\frac{\partial P}{\partial t}dt=\frac{1}{\rho}dp$

$$=\frac{1}{\rho}\left[\frac{\partial p}{\partial x_1}dx_1+\frac{\partial p}{\partial x_2}dx_2+\frac{\partial p}{\partial x_3}dx_3+\frac{\partial p}{\partial t}dt\right]$$

where $P=P(x_1,x_2,x_3,t),\ p=p(x_1,x_2,x_3,t)$

Equating the coefficients of differntial dx_i, we get

$$\frac{\partial P}{\partial x_1}=\frac{1}{\rho}\frac{\partial p}{\partial x_1},\ \frac{\partial P}{\partial x_2}=\frac{1}{\rho}\frac{\partial p}{\partial x_2},\ \frac{\partial P}{\partial x_3}=\frac{1}{\rho}\frac{\partial p}{\partial x_3},\ \frac{\partial P}{\partial t}=\frac{1}{\rho}\frac{\partial p}{\partial t}$$

$$\Rightarrow \vec{\nabla}P=\frac{1}{\rho}\vec{\nabla}p \quad \text{.....................} (30)$$

Again when external body forces are conservative, then

$$\vec{F}=-\vec{\nabla}\Omega$$

and existence of velocity potential implies the motion is irrotational and hence

$$\vec{V}=-\vec{\nabla}\varphi.$$

Hence Euler's equation becomes

$$\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\vec{\nabla})\vec{V}=-\vec{\nabla}\Omega-\vec{\nabla}P \quad \text{................} (31)$$

or, $\frac{\partial}{\partial t}(-\vec{\nabla}\varphi)+\frac{1}{2}\vec{\nabla}V^2-\vec{V}\times(\vec{\nabla}\times\vec{V})=-\vec{\nabla}\Omega-\vec{\nabla}P$

$\left(\because \vec{\nabla}(\vec{V}\cdot\vec{V})=2\left[\vec{V}\times(\vec{\nabla}\times\vec{V})+(\vec{V}\cdot\vec{\nabla})\vec{V}\right]\right)$

or, $\vec{\nabla}\left(\Omega+P+\frac{1}{2}V^2-\frac{\partial\varphi}{\partial t}\right)=\vec{0}\ \left(\because \vec{\nabla}\times\vec{V}=\vec{\nabla}\times(-\vec{\nabla}\varphi)=0\right)$

Taking dot product by $d\vec{r}\left(=\hat{i}dx_1+\hat{j}dx_2+\hat{k}dx_3\right)$ and noting that $d\vec{r}\cdot\vec{\nabla}=d$, then above equation becomes

$$d\left(\Omega+P+\frac{1}{2}V^2-\frac{\partial\varphi}{\partial t}\right)=0$$

Integrating, $\Omega + P + \frac{1}{2}V^2 - \frac{\partial\varphi}{\partial t}$ = constant of Int. = $F(t)$, say

i.e., $\Omega + \int\frac{dp}{\rho} + \frac{1}{2}V^2 - \frac{\partial\varphi}{\partial t} = F(t)$ .... (32)

which is known as *Bernoulli's equation for unsteady irrotational motion*. This is also known as *pressure equation.*

Note.1. If the fluid is incompressible, then ρ is constant and hence (32) becomes

$$\Omega + \frac{p}{\rho} + \frac{1}{2}V^2 - \frac{\partial\varphi}{\partial t} = F(t).$$

Note.2. If the fluid motion is steady, then $\frac{\partial\varphi}{\partial t} = 0$ and hence (32) becomes

$$\Omega + \int\frac{dp}{\rho} + \frac{1}{2}V^2 = F(t) = C = \text{absolute constant.}$$

i.e., $\Omega + \int\frac{dp}{\rho} + \frac{1}{2}V^2 = C$ .... (33)

This is known as Bernoulli's equation for steady motion.

If ρ = constant, then we have from above

$\Omega + \frac{p}{\rho} + \frac{1}{2}V^2 = C$ .... (34)

Note.3. Equations (32) and (33) are known as the integral of Euler's equation of motion for unsteady motion (irrotational) and for steady rotational flow.

5.22 Integrals of Lagrange's Equations of Motion:

The Lagrange's equations of motion are, from (28),

$$\left(F_i - \frac{\partial^2 x_i}{\partial t^2}\right)\frac{\partial x_i}{\partial X_j} = \frac{1}{\rho}\frac{\partial p_i}{\partial X_j} (j = 1,2,3)$$

and equation of continuity is

$$\rho_0 = \rho\frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)}.$$

to determine five unknown quantities x_1, x_2, x_3, p, ρ each depending on X_1, X_2, X_3, t. Now we assume that p is a function of ρ alone i.e., $p = f(\rho)$, known as *barotropic fluid*, so that $\int \frac{dp}{\rho}$ exists. We introduce pressure potential P as

$$P = \int_0^p \frac{dp}{\rho}$$

$$\frac{dP}{dp} = \frac{1}{\rho}$$

i.e. $$dP = \frac{1}{\rho} dp$$

or, $$\frac{\partial P}{\partial X_i} dX_i + \frac{\partial P}{\partial t} dt = \frac{1}{\rho}\left(\frac{\partial p}{\partial X_i} dX_i + \frac{\partial p}{\partial t} dt\right)$$

which $\Rightarrow \frac{\partial P}{\partial X_i} = \frac{1}{\rho}\frac{\partial p}{\partial X_i}$

When the external body forces are conservative then

$$\bar{F} = -\vec{\nabla}\Omega$$

i.e., $$F_i = -\frac{\partial \Omega}{\partial x_i}$$

where $\Omega = \Omega(x_1, x_2, x_3)$ is scalar potential function.

Hence, with the help of above relations, the Lagrange's equations of motion takes the form

$$\frac{\partial^2 x_i}{\partial t^2} \cdot \frac{\partial x_i}{\partial X_j} = F_i \frac{\partial x_i}{\partial X_j} - \frac{1}{\rho}\frac{\partial p}{\partial X_j}$$

$$= -\frac{\partial \Omega}{\partial x_i} \cdot \frac{\partial x_i}{\partial X_j} - \frac{\partial P}{\partial X_j}$$

$$= -\frac{\partial \Omega}{\partial X_j} - \frac{\partial P}{\partial X_j}$$

$$= -\frac{\partial}{\partial X_j}(\Omega + P)$$

or, $$\frac{\partial^2 x_i}{\partial t^2} \cdot \frac{\partial x_i}{\partial X_j} = -\frac{\partial Q}{\partial X_j}, j = 1,2,3 \quad \text{.........................} (35)$$

where $Q = \Omega + P = \Omega + \int_0^p \frac{dp}{\rho}$ .... (36)

which is Lagrange's equations of motion for barotropic fluid in the conservative fluid of forces.

To obtain the integral of these equations of motion, we eliminate Q from (35) with the help of the idea

$$\frac{\partial}{\partial X_k}\left(\frac{\partial Q}{\partial X_j}\right) = \frac{\partial}{\partial X_j}\left(\frac{\partial Q}{\partial X_k}\right) (j, k = 1, 2, 3; j \neq k)$$

i.e., $$\frac{\partial}{\partial X_k}\left(\frac{\partial^2 x_i}{\partial t^2} \cdot \frac{\partial x_i}{\partial X_j}\right) = \frac{\partial}{\partial X_j}\left(\frac{\partial^2 x_i}{\partial t^2} \cdot \frac{\partial x_i}{\partial X_k}\right)$$

or, $$\frac{\partial \ddot{x}_i}{\partial X_k} \cdot \frac{\partial x_i}{\partial X_j} = \frac{\partial \ddot{x}_i}{\partial X_j} \cdot \frac{\partial x_i}{\partial X_k} \text{ where } \dot{x}_i = \frac{\partial x_i}{\partial t}, \ddot{x}_i = \frac{\partial^2 x_i}{\partial t^2}$$

or, $$\frac{\partial}{\partial t}\left[\frac{\partial \dot{x}_i}{\partial X_k} \cdot \frac{\partial x_i}{\partial X_j} - \frac{\partial \dot{x}_i}{\partial X_j} \cdot \frac{\partial x_i}{\partial X_k}\right] = 0$$

or, $$\frac{\partial}{\partial t}\left[\frac{\partial v_i}{\partial X_k} \cdot \frac{\partial x_i}{\partial X_j} - \frac{\partial v_i}{\partial X_j} \cdot \frac{\partial x_i}{\partial X_k}\right] = 0 \text{ where } v_i = \dot{x}_i = \frac{\partial x_i}{\partial t}$$

Integrating, $$\frac{\partial v_i}{\partial X_k} \cdot \frac{\partial x_i}{\partial X_j} - \frac{\partial v_i}{\partial X_j} \cdot \frac{\partial x_i}{\partial X_k} = \text{constant} = C_0$$

Initially, i.e., at $t = 0, x_1 = X_1, x_2 = X_2, x_3 = X_3; v_1 = V_1, v_2 = V_2, v_3 = V_3$

$$\frac{\partial x_i}{\partial X_j} = 1, \text{ for } i = j \text{ and } \frac{\partial x_i}{\partial X_j} = 0 \text{ for } i \neq j.$$

$$\therefore C_0 = \frac{\partial V_i}{\partial X_k} \cdot \frac{\partial X_i}{\partial X_j} - \frac{\partial V_i}{\partial X_j} \cdot \frac{\partial X_i}{\partial X_k}$$

Hence, we have

$$\frac{\partial v_i}{\partial X_k} \cdot \frac{\partial x_i}{\partial X_j} - \frac{\partial v_i}{\partial X_j} \cdot \frac{\partial x_i}{\partial X_k} = \frac{\partial V_i}{\partial X_k}\delta_{ij} - \frac{\partial V_i}{\partial X_j}\delta_{ik}$$

$$= \frac{\partial V_j}{\partial X_k} - \frac{\partial V_k}{\partial X_j} (\text{for } j, k = 1, 2, 3; j \neq k) \quad \text{.......................} (37)$$

These are integrals of Lagrange's equation of motion.

5.23 Cauchy's Integrals:

To obtain Cauchy's integrals we consider the Lagrange's hydrodynamical equations of motion

$$\frac{\partial v_i}{\partial X_k} \cdot \frac{\partial x_i}{\partial X_j} - \frac{\partial v_i}{\partial X_j} \cdot \frac{\partial x_i}{\partial X_k} = \frac{\partial V_j}{\partial X_k} - \frac{\partial V_k}{\partial X_j} (j, k = 1, 2, 3, j \neq k)$$

Taking $k = 2, j = 3$ then we have

$$\left(\frac{\partial v_1}{\partial X_2} \cdot \frac{\partial x_1}{\partial X_3} - \frac{\partial v_1}{\partial X_3} \cdot \frac{\partial x_1}{\partial X_2}\right) + \left(\frac{\partial v_2}{\partial X_2} \cdot \frac{\partial x_2}{\partial X_3} - \frac{\partial v_2}{\partial X_3} \cdot \frac{\partial x_2}{\partial X_2}\right) + \left(\frac{\partial v_3}{\partial X_2} \cdot \frac{\partial x_3}{\partial X_3} - \frac{\partial v_3}{\partial X_3} \cdot \frac{\partial x_3}{\partial X_2}\right)$$

$$= \frac{\partial V_3}{\partial X_2} - \frac{\partial V_2}{\partial X_3} \quad \text{.......................} (38)$$

Now,

$$\frac{\partial v_1}{\partial X_2} \cdot \frac{\partial x_1}{\partial X_3} - \frac{\partial v_1}{\partial X_3} \cdot \frac{\partial x_1}{\partial X_2} = \frac{\partial v_1}{\partial x_i} \cdot \frac{\partial x_i}{\partial X_2} \cdot \frac{\partial x_1}{\partial X_3} - \frac{\partial v_1}{\partial x_i} \cdot \frac{\partial x_i}{\partial X_3} \cdot \frac{\partial x_1}{\partial X_2}$$

$$= \frac{\partial v_1}{\partial x_3}\left(\frac{\partial x_3}{\partial X_2} \cdot \frac{\partial x_1}{\partial X_3} - \frac{\partial x_3}{\partial X_3} \cdot \frac{\partial x_1}{\partial X_2}\right) - \frac{\partial v_1}{\partial x_2}\left(\frac{\partial x_2}{\partial X_3} \cdot \frac{\partial x_1}{\partial X_2} - \frac{\partial x_2}{\partial X_2} \cdot \frac{\partial x_1}{\partial X_3}\right)$$

$$= \frac{\partial v_1}{\partial x_3} \cdot \frac{\partial(x_3, x_1)}{\partial(X_2, X_3)} - \frac{\partial v_1}{\partial x_2} \cdot \frac{\partial(x_1, x_2)}{\partial(X_2, X_3)}$$

Similarly, $$\frac{\partial v_2}{\partial X_2} \cdot \frac{\partial x_2}{\partial X_3} - \frac{\partial v_2}{\partial X_3} \cdot \frac{\partial x_2}{\partial X_2} = \frac{\partial v_2}{\partial x_1} \cdot \frac{\partial(x_1; x_2)}{\partial(X_2, X_3)} - \frac{\partial v_2}{\partial x_3} \cdot \frac{\partial(x_2, x_3)}{\partial(X_2, X_3)}$$

and $$\frac{\partial v_3}{\partial X_2} \cdot \frac{\partial x_3}{\partial X_3} - \frac{\partial v_3}{\partial X_3} \cdot \frac{\partial x_3}{\partial X_2} = \frac{\partial v_3}{\partial x_2} \cdot \frac{\partial(x_2, x_3)}{\partial(X_2, X_3)} - \frac{\partial v_3}{\partial x_1} \cdot \frac{\partial(x_3, x_1)}{\partial(X_2, X_3)}$$

Using above results in (38), we get

$$\left(\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}\right)\frac{\partial(x_2,x_3)}{\partial(X_2,X_3)}+\left(\frac{\partial v_1}{\partial x_3}-\frac{\partial v_3}{\partial x_1}\right)\frac{\partial(x_3,x_1)}{\partial(X_2,X_3)}+\left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right)\frac{\partial(x_1,x_2)}{\partial(X_2,X_3)}$$

$$\frac{\partial V_3}{\partial X_2}-\frac{\partial V_2}{\partial X_3} \qquad \text{.......................} (39)$$

If W_1, W_2, W_3 be the components of the vorticity vector $\vec{W}=\vec{\nabla}\times\vec{V}$ and W_{01}, W_{02}, W_{03} be their initial values, then equation (39) can be written as

$$W_1\frac{\partial(x_2,x_3)}{\partial(X_2,X_3)}+W_2\frac{\partial(x_3,x_1)}{\partial(X_2,X_3)}+W_3\frac{\partial(x_1,x_2)}{\partial(X_2,X_3)}=W_{01} \qquad \text{.....................} (40)$$

Similarly from other two equations (37), we get

$$W_1\frac{\partial(x_2,x_3)}{\partial(X_3,X_1)}+W_2\frac{\partial(x_3,x_1)}{\partial(X_3,X_1)}+W_3\frac{\partial(x_1,x_2)}{\partial(X_3,X_1)}=W_{02} \qquad \text{..........................} (41)$$

and $$W_1\frac{\partial(x_2,x_3)}{\partial(X_1,X_2)}+W_2\frac{\partial(x_3,x_1)}{\partial(X_1,X_2)}+W_3\frac{\partial(x_1,x_2)}{\partial(X_1,X_2)}=W_{03} \qquad \text{..........................} (42)$$

Multiplying (40), (41), (42) by $\frac{\partial x_1}{\partial X_1}, \frac{\partial x_1}{\partial X_2}, \frac{\partial x_1}{\partial X_3}$ respectively and adding we get

$$W_1\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}+W_2\frac{\partial(x_1,x_3,x_1)}{\partial(X_1,X_2,X_3)}+W_3\frac{\partial(x_1,x_1,x_3)}{\partial(X_1,X_2,X_3)}=W_{01}\frac{\partial x_1}{\partial X_1}+W_{02}\frac{\partial x_1}{\partial X_2}+W_{03}\frac{\partial x_1}{\partial X_3}$$

or, $$W_1\frac{\rho_0}{\rho}=W_{01}\frac{\partial x_1}{\partial X_1}+W_{02}\frac{\partial x_1}{\partial X_2}+W_{03}\frac{\partial x_1}{\partial X_3}\left(\because \rho_0=\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}\rho\right)$$

or, $$\frac{W_1}{\rho}=\frac{W_{01}}{\rho_0}\frac{\partial x_1}{\partial X_1}+\frac{W_{02}}{\rho_0}\frac{\partial x_1}{\partial X_2}+\frac{W_{03}}{\rho_0}\frac{\partial x_1}{\partial X_3} \qquad \text{........................} (43)$$

Similarly multiplying (40), (41), (42) by $\frac{\partial x_2}{\partial X_1}, \frac{\partial x_2}{\partial X_2}, \frac{\partial x_2}{\partial X_3}$ respectively and $\frac{\partial x_3}{\partial X_1}, \frac{\partial x_3}{\partial X_2}, \frac{\partial x_3}{\partial X_3}$ respectively

then adding in each case we get

$$\frac{W_2}{\rho} = \frac{W_{01}}{\rho_0}\frac{\partial x_2}{\partial X_1} + \frac{W_{02}}{\rho_0}\frac{\partial x_2}{\partial X_2} + \frac{W_{03}}{\rho_0}\frac{\partial x_2}{\partial X_3} \quad \text{......................} (44)$$

and $$\frac{W_3}{\rho} = \frac{W_{01}}{\rho_0}\frac{\partial x_3}{\partial X_1} + \frac{W_{02}}{\rho_0}\frac{\partial x_3}{\partial X_2} + \frac{W_{03}}{\rho_0}\frac{\partial x_3}{\partial X_3} \quad \text{......................} (45)$$

The equations (43), (44) and (45) are called Cauchy's integrals of Lagrange's equation of motion expressed entirely in terms of vorticity. The vector form of these equations is

$$\frac{\vec{W}}{\rho} = \left(\frac{W_{01}}{\rho_0}\frac{\partial}{\partial X_1} + \frac{W_{02}}{\rho_0}\frac{\partial}{\partial X_2} + \frac{W_{03}}{\rho_0}\frac{\partial}{\partial X_3}\right)\vec{r}, \quad \vec{r} = x_1\hat{i} + x_2\hat{j} + x_3\hat{j}$$

i.e., $$\frac{\vec{W}}{\rho} = \left(\frac{\vec{W}}{\rho}\cdot\vec{\nabla}\right)_0 \vec{r}$$

Note.1. *Permanence of irrotational motion:*

If the motion of the fluid is initially irrotational, then $W_{01} = W_{02} = W_{03} = 0$. It follows from (43), (44), (45) that $W_1 = W_2 = W_3 = 0$ at any time t.

Hence the motion of the fluid, once irrotational is always irrotational, provides the external forces are conservative and pressure p is a function of density ρ alone.

Note.2. *Helmholtz's vorticity equation from Cauchy's Integral:*

Diff. (43) w.r.t. time t

$$\frac{d}{dt}\left(\frac{W_1}{\rho}\right) = \frac{W_{01}}{\rho_0}\frac{d}{dt}\left(\frac{\partial x_1}{\partial X_1}\right) + \frac{W_{02}}{\rho_0}\frac{d}{dt}\left(\frac{\partial x_1}{\partial X_2}\right) + \frac{W_{03}}{\rho_0}\frac{d}{dt}\left(\frac{\partial x_1}{\partial X_3}\right)$$

$$= \frac{W_{01}}{\rho_0}\frac{d}{dX_1}\left(\frac{dx_1}{dt}\right) + \frac{W_{02}}{\rho_0}\frac{d}{dX_2}\left(\frac{dx_1}{dt}\right) + \frac{W_{03}}{\rho_0}\frac{d}{dX_3}\left(\frac{dx_1}{dt}\right)$$

$$\therefore \frac{d}{dt}\left(\frac{W_1}{\rho}\right) = \frac{W_{01}}{\rho_0}\frac{dv_1}{dX_1} + \frac{W_{02}}{\rho_0}\frac{dv_1}{dX_2} + \frac{W_{03}}{\rho_0}\frac{dv_1}{dX_3} \quad \text{.........} (46)$$

Now, $\frac{W_1}{\rho}\frac{\partial v_1}{\partial x_1} + \frac{W_2}{\rho}\frac{\partial v_1}{\partial x_2} + \frac{W_3}{\rho}\frac{\partial v_1}{\partial x_3}$

$$= \frac{W_{01}}{\rho_0}\left(\frac{\partial v_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial X_1} + \frac{\partial v_1}{\partial x_2}\cdot\frac{\partial x_2}{\partial X_1} + \frac{\partial v_1}{\partial x_3}\cdot\frac{\partial x_3}{\partial X_1}\right) + \frac{W_{02}}{\rho_0}\left(\frac{\partial v_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial X_1} + \frac{\partial v_1}{\partial x_2}\cdot\frac{\partial x_2}{\partial X_2} + \frac{\partial v_1}{\partial x_3}\cdot\frac{\partial x_3}{\partial X_2}\right)$$

$$+ \frac{W_{03}}{\rho_0}\left(\frac{\partial v_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial X_3} + \frac{\partial v_1}{\partial x_2}\cdot\frac{\partial x_2}{\partial X_3} + \frac{\partial v_1}{\partial x_3}\cdot\frac{\partial x_3}{\partial X_3}\right)$$

$$\therefore \frac{W_1}{\rho}\frac{\partial v_1}{\partial x_1} + \frac{W_2}{\rho}\frac{\partial v_1}{\partial x_2} + \frac{W_3}{\rho}\frac{\partial v_1}{\partial x_3} = \frac{W_{01}}{\rho_0}\frac{\partial v_1}{\partial X_1} + \frac{W_{02}}{\rho_0}\frac{\partial v_1}{\partial X_2} + \frac{W_{03}}{\rho_0}\frac{\partial v_1}{\partial X_3} \quad \text{................} (47)$$

From (46) and (47), weg get

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{W_1}{\rho}\right) &= \frac{W_1}{\rho}\frac{\partial v_1}{\partial x_1} + \frac{W_2}{\rho}\frac{\partial v_1}{\partial x_2} + \frac{W_3}{\rho}\frac{\partial v_1}{\partial x_3} \\ \text{Similarly,}\quad \frac{d}{dt}\left(\frac{W_2}{\rho}\right) &= \frac{W_1}{\rho}\frac{\partial v_2}{\partial x_1} + \frac{W_2}{\rho}\frac{\partial v_2}{\partial x_2} + \frac{W_3}{\rho}\frac{\partial v_2}{\partial x_3} \\ \text{and}\quad \frac{d}{dt}\left(\frac{W_3}{\rho}\right) &= \frac{W_1}{\rho}\frac{\partial v_3}{\partial x_1} + \frac{W_2}{\rho}\frac{\partial v_3}{\partial x_2} + \frac{W_3}{\rho}\frac{\partial v_3}{\partial x_3} \end{aligned}\right\} \quad \text{...........................} (48)$$

which are equivalent to single vector equation

$$\frac{d}{dt}\left(\frac{\vec{W}}{\rho}\right) = \left(\frac{\vec{W}}{\rho}\cdot\vec{\nabla}\right)\vec{V}$$

or, $$\rho\frac{d}{dt}\left(\frac{\vec{W}}{\rho}\right) = \left(\vec{W}\cdot\vec{\nabla}\right)\vec{V}$$

This is known as Helmholtz vorticity equation for perfect fluid.

5.24 Unit Summary:

In this module we have discussed different methods to understand the behaviour of the perfect fluid, its motion, its integral of equation of motion etc.

5.25 Worked Out Examples

Ex.1. *Find the stream lines and paths of the particle for the*

i) *Velocity field* $v_1 = \dfrac{x_1}{1+t}, v_2 = x_2, v_3 = 0;$

ii) *Velocity field* $v_1 = \dfrac{x_1^2}{1+t^2}, v_2 = x_2^2, v_3 = 0.$

Ans. To determine the stream lines, we have the differential equation at a given instant of time t as

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3}$$

i.e. $$\frac{dx_1}{\frac{x_1}{1+t}} = \frac{dx_2}{x_2} = \frac{dx_3}{0}$$

From 3rd ratio, $dx_3 = 0 \Rightarrow dx_3 =$ consant $= a$

and from 1st and 2nd ratios we get

$$\left(\frac{1+t}{x_1}\right)dx_1 = \frac{dx_2}{x_2}$$

$$\Rightarrow \quad (1+t)\log_e x_1 = \log_e x_2 + \log_e b, b \text{ being constant}$$

$$\Rightarrow \quad x_1^{(1+t)} = bx_2$$

Hence, $x_1^{(1+t)} = bx_2, x_3 = a$ represent the equation of the stream lines.

To determine path lines we have the solutions of the differential equations

$$\frac{dx_1}{dt} = v_1, \frac{dx_2}{dt} = v_2, \frac{dx_3}{dt} = v_3$$

This $\Rightarrow$ that $\dfrac{dx_1}{dt} = \dfrac{x_1}{1+t} \Rightarrow \dfrac{dx_1}{x_1} = \dfrac{dt}{1+t}$

i.e. $\log_e x_1 = \log_e(1+t) + \log_e c$, c being constant

$$\Rightarrow x_1 = c(1+t)$$

$$\frac{dx_2}{dt} = x_2 \Rightarrow \frac{dx_2}{x_2} = dt \Rightarrow \log_e x_2 = t + \log_e d$$

$$x_2 = de^t$$

and $\frac{dx_3}{dt} = 0 \Rightarrow x_3 =$ constant $= p$, say

Hence $x_2 = de^{\left(\frac{x_1}{c}-1\right)}, x_3 = p$ represent the equations of path lines.

ii) The stream lines are given by the differntial equation

$$\frac{dx_1}{\frac{x_1^2}{1+t^2}} = \frac{dx_2}{x_2^2} = \frac{dx_3}{0}$$

From 3rd ratio we get $x_3 =$ constant $= c_1$, say

and from 1st, 2nd ratios we get

$$(1+t^2)\frac{dx_1}{x_1^2} = \frac{dx_2}{x_2^2}$$

or, $(1+t^2)\frac{1}{x_1} - \frac{1}{x_2} = c_2$, c_2 being constant.

Hence the stream lines are given by $(1+t^2)\frac{1}{x_1} - \frac{1}{x_2} = c_2 \;\; x_3 = c_1$.

The path lines are given by

$$\frac{dx_1}{dt} = \frac{x_1^2}{(1+t^2)} \Rightarrow -\frac{1}{x_1} + x_3 = \tan^{-1}(t)$$

$$\frac{dx_2}{dt} = x_2^2 \Rightarrow -\frac{1}{x_2} + x_4 = t$$

$$\frac{dx_3}{dt} = 0 \Rightarrow x_3 = c_5$$

where c_3, c_4, c_5 are constants. Eliminating t from above three relations we get

$$\tan\left(c_3 - \frac{1}{x_1}\right) = c_4 - \frac{1}{x_2}, x_3 = c_5$$

which is the equation of the path line.

Ex.2. *If the velocity of an incompressible fluid at the point* (x, y, z) *is given by* $\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2 - r^2}{r^5}\right)$*; prove that the liquid motion is possible and the velocity potential is* $\frac{\cos\theta}{r^2}$*. Also determine the stream lines.*

Ans. If (u, v, w) be the velocity components then

$$u = v_1 = \frac{3xz}{r^5}, v = v_2 = \frac{3yz}{r^5}, w = v_3 = \frac{3z^2 - r^2}{r^5} \text{ where } r^2 = x^2 + y^2 + z^2.$$

Now the liquid motion is to be possible if it satisfies the equation of continuity; i.e. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ for incompressible fluid.

Also $\frac{\partial u}{\partial x} = \frac{3z}{r^5} - \frac{15x^2}{r^7} = \frac{3z}{r^{10}}\left(r^5 - 5r^3x^2\right),$

$$\frac{\partial v}{\partial y} = \frac{3z}{r^{10}}\left(r^5 - 5r^3x^2\right)$$

$$\frac{\partial w}{\partial z} = \frac{1}{r^{10}}\left[(6z - 2z)r^5 - 5r^3\left(3z^2 - r^2\right)z\right].$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{3z}{r^{10}}\left[2r^5 - 5r^3\left(r^2 - z^2\right)\right] + \frac{1}{r^{10}}\left[9zr^5 - 15r^3z^3\right]$$

$$= 0$$

which implies that the liquid motion is possible.

Let φ be the velocity potential. Then

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz = -udx - vdy - wdz$$

$$= -\frac{1}{r^5}\left[3xz\,dx + 3yz\,dy + \left(3z^2 - r^2\right)dz\right]$$

$$= -\frac{1}{r^5}\left[3z\left(xdx + ydy + zdz\right) - r^2dz\right]$$

$$= -\frac{1}{r^5}\left[\frac{3z}{2}d\left(r^2\right) - r^2dz\right]$$

$$= -\frac{3z}{r^4}dr + \frac{dz}{r^3}$$

$$= d\left(\frac{z}{r^3}\right)$$

Int. $\varphi = \frac{z}{r^3}$, omiting constant

$$= \frac{r\cos\theta}{r^3} = \frac{\cos\theta}{r^2}$$

Again stream lines are given by the solutions of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

or, $$\frac{dx}{3xz} = \frac{dy}{3yz} = \frac{dz}{3z^2 - r^2} = \frac{xdx + ydy + zdz}{3z\left(x^2 + y^2 + z^2\right) - r^2 z}$$

From 1st and 2nd ratios, we get

$$\frac{dx}{x} + \frac{dy}{y} \Rightarrow x = c_1 y,\ c_1 \text{ is constant.}$$

From 1st and 4th ratios, we get

$$\frac{dx}{3x} = \frac{xdx + ydy + zdz}{2r^2} = \frac{d\left(r^2\right)}{4r^2}$$

$$4\log_e x = 3\log_e r^2 + \log c_2$$

$$x^4 = c_2 r^6 = c_2\left(x^2 + y^2 + z^2\right)^3$$

Hence $x = c_1 y, x^4 = c_2\left(x^2 + y^2 + z^2\right)^3$ represent the equations of stream lines.

Ex.3. *Show that* $u = -\frac{2xyz}{\left(x^2 + y^2\right)^2}, v = \frac{\left(x^2 - y^2\right)z}{\left(x^2 + y^2\right)^2}, w = \frac{y}{x^2 + y^2}$ *are the velocity-components of a possible liquid motion. Is this motion irrotational?*

Ans. To show that the motion is possible, we have to show that the equation of continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ is satisfied.

Now, $\dfrac{\partial u}{\partial x} = -\dfrac{2yz\left(y^2 - 3x^2\right)}{\left(x^2 + y^2\right)^3}$

$$\frac{\partial v}{\partial y} = -\frac{2yz\left(3x^2 - y^2\right)}{\left(x^2 + y^2\right)^3}, \quad \frac{\partial w}{\partial z} = 0$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{2yz}{\left(x^2 + y^2\right)^3}\left[\left(3x^2 - y^2\right) + \left(y^2 - 3x^2\right) + 0\right] = 0.$$

Hence the liquid motion is possible.

Again fluid motion to be irrotational if $\vec{\nabla} \times \vec{V} = \vec{0}$

i.e., if $\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x} = 0,\ \dfrac{\partial v}{\partial x} - \dfrac{\partial w}{\partial y} = 0,\ \dfrac{\partial w}{\partial x} - \dfrac{\partial u}{\partial y} = 0.$

Now, $\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x} = -\dfrac{2xz\left(x^2 - 3y^2\right)}{\left(x^2 + y^2\right)^3} - \dfrac{2xz\left(3y^2 - x^2\right)}{\left(x^2 + y^2\right)^3} = 0$

$$\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = \frac{\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2} - \frac{\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2} = 0$$

$$\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = -\frac{2xy}{\left(x^2 + y^2\right)^2} + \frac{2xy}{\left(x^2 + y^2\right)^2} = 0$$

which shows that the motion is irrotational.

Ex.4. *For the velocity field* $v_1 = \dfrac{3x_1^2 - r^2}{r^5},\ v_2 = \dfrac{3x_1x_2}{r^5},\ v_3 = \dfrac{3x_1x_3}{r^5}$

where $\left(x_1^2 + x_2^2 + x_3^2\right)^3 = c\left(x_2^2 + x_3^2\right)^2$*, by the planes passing through* Ox_1*. Also find the velocity potential.*

Ans. Now $\dfrac{\partial v_1}{\partial x_1} = \dfrac{3x_1\left(3r^5 - 5r^3x_1^2\right)}{r^{10}},\ \dfrac{\partial v_2}{\partial x_2} = \dfrac{3x_1\left(r^5 - 5x_2^2r^3\right)}{r^{10}},$

$$\frac{\partial v_3}{\partial x_3} = \frac{3x_1\left(r^5 - 5x_3^2r^3\right)}{r^{10}}.$$

$$\therefore \frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}=\frac{3x_1}{r^{10}}\left[5r^5-5r^3\left(x_1^2+x_2^2+x_3^2\right)\right]=\frac{3x_1}{r^{10}}\left[5r^5-5r^5\right]$$

$$=\frac{3x_1}{r^{10}}\cdot 0=0.$$

Since $\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}=0$, so the equation of continuity is satisfied and hence liquid motion is possible.

ii) The fluid motion is said to be irrotational if $\vec{\nabla}\times\vec{V}=\vec{0}$,

i.e., $$\vec{\nabla}\times\vec{V}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}\\ v_1 & v_2 & v_3\end{vmatrix}=\vec{0}$$

i.e., if $\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}=0,\ \frac{\partial v_1}{\partial x_3}-\frac{\partial v_3}{\partial x_1}=0,\ \frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}=0$

Now, $\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}=3x_1x_3\left(-\frac{5}{r^6}\right)\cdot\frac{x_2}{r}-3x_1x_2\left(-\frac{5}{r^6}\right)\frac{x_3}{r}$ where $\frac{\partial r}{\partial x_2}=\frac{x_2}{r}$ etc.

$$=-\frac{15\,x_1x_2x_3}{r^7}+\frac{15\,x_1x_2x_3}{r^7}$$

$$=0$$

Similarly, $\frac{\partial v_1}{\partial x_3}-\frac{\partial v_3}{\partial x_1}=0$ and $\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}=0$

Hence $\vec{\nabla}\times\vec{V}=\vec{0}$. Therefore the fluid motion is irrotational.

iii) The stream lines are the solutions of the differential equation

$$\frac{dx_1}{v_1}=\frac{dx_2}{v_2}=\frac{dx_3}{v_3}$$

or, $$\frac{dx_1}{3x_1^2-r^2}=\frac{dx_2}{3x_1x_2}=\frac{dx_3}{3x_1x_3}=\frac{x_1dx_1+x_2dx_2+x_3dx_3}{x_1\left(3r^2-r^2\right)}$$

$$=\frac{x_2dx_2+x_3dx_3}{3x_1\left(x_2^2+x_3^2\right)}$$

From 2nd and 3rd ratios we get

$$\frac{dx_2}{x_2} = \frac{dx_3}{x_3}$$

$\Rightarrow x_2 = c_1 x_3$, c_1 being constant.

Again from 4th and 5th ratios, we get

$$\frac{x_1 dx_1 + x_2 dx_2 + x_3 dx_3}{2r^2} = \frac{x_2 dx_2 + x_3 dx_3}{3\left(x_2^2 + x_3^2\right)}$$

or, $$\frac{d\left(x_1^2 + x_2^2 + x_3^2\right)}{2\left(x_1^2 + x_2^2 + x_3^2\right)} = \frac{d\left(x_2^2 + x_3^2\right)}{3\left(x_2^2 + x_3^2\right)}$$

Int. $\frac{1}{2}\log_e\left(x_1^2 + x_2^2 + x_3^2\right) = \frac{1}{3}\log_e\left(x_2^2 + x_3^2\right) + \frac{1}{6}\log c_2$, c_2 being constant.

or, $$\left(x_1^2 + x_2^2 + x_3^2\right)^3 = c_2\left(x_2^2 + x_3^2\right)^2$$

Hence the stream lines are the intersection of the surfaces, $\left(x_1^2 + x_2^2 + x_3^2\right)^3 = c_2\left(x_2^2 + x_3^2\right)^2$ by the plane $x_2 = c_1 x_3$ which passing through Ox_1.

Let φ be the velocity potential. Then we have

$$d\varphi = \frac{\partial\varphi}{\partial x_1}dx_1 + \frac{\partial\varphi}{\partial x_2}dx_2 + \frac{\partial\varphi}{\partial x_3}dx_3 = -v_1 dx_1 - v_2 dx_2 - v_3 dx_3$$

$$= -\left\{\frac{3x_1^2 - r^2}{r^5}dx_1 + \frac{3x_1 x_2}{r^5}dx_2 + \frac{3x_1 x_3}{r^5}dx_3\right\}$$

$$= -\frac{1}{r^5}\left\{3x_1\left(x_1 dx_1 + x_2 dx_2 + x_3 dx_3\right) - r^2 dx_1\right\}$$

$$= -\frac{1}{r^5}\left\{\frac{3x_1}{2}d\left(x_1^2 + x_2^2 + x_3^2\right) - r^2 dx_1\right\}$$

$$= -\frac{1}{r^5}\left\{\frac{3x_1}{2}d\left(r^2\right) - r^2 dx_1\right\}$$

$$= -\left\{\frac{3x_1}{2r^5}d\left(r^2\right) - \frac{1}{r^3}dx_1\right\}$$

$$= \frac{dx_1}{r^3} - \frac{3x_1}{r^4} dr$$

$$= d\left(\frac{x_1}{r^3}\right)$$

$$\Rightarrow \varphi = \frac{x_1}{r^3} = \frac{x_1}{\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}}, \text{omiting constant,}$$

which is the required velocity potential.

Ex.5. *Given* $v_1 = -wx_2, v_2 = wx_1, v_3 = 0$; *show that the surfaces intersecting the stream lines orthogonally exist and are the planes through z-axis, although the velocity potential does not exist.*

Ans. Now $\frac{\partial v_1}{\partial x_1} = 0, \frac{\partial v_2}{\partial x_2} = 0$ and $\frac{\partial v_3}{\partial x_3} = 0$

$\therefore \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0$ is satisfied and hence the fluid motion is possible.

To show that the surfaces orthogonal to stream lines are planes through z-axis.

The required surfaces are solutions of

$$v_1 dx_1 + v_2 dx_2 + v_3 dx_3 = 0$$

or, $\quad -wx_2 dx_1 + wx_1 dx_2 + 0dx_3 = 0$

or, $\quad \frac{dx_1}{x_1} - \frac{dx_2}{x_2} = 0$

Int., $x_1 = cx_2$, c being constant,

which is a plane through z-axis.

To show that velocity potential φ does not exist.

Now, $\quad d\varphi = \frac{\partial \varphi}{\partial x_1} dx_1 + \frac{\partial \varphi}{\partial x_2} dx_2 + \frac{\partial \varphi}{\partial x_3} dx_3$

$$= -(v_1 dx_1 + v_2 dx_2 + v_3 dx_3)$$

$$= -(-wx_2 dx_1 + wx_1 dx_2 + Odx_3)$$

$$= wx_2 dx_2 - wx_1 dx_2$$

$$= Mdx_1 + Ndx_2, \text{ say}$$

where $M = wx_2$, $N = -wx_1$.

Here $\frac{\partial M}{\partial x_2} = w$, $\frac{\partial N}{\partial x_1} = -w$

$$\therefore \frac{\partial M}{\partial x_2} \neq \frac{\partial N}{\partial x_1}$$

which implies that the above differential equation is not exact and hence $d\varphi = wx_2 dx_1 - wx_1 dx_2$ cannot be integrated so that φ does not exist.

Ex.6. *Show that in the motion of a fluid in two dimensions if the current co-ordinates (x_1, x_2) are expressible in terms of initial co-ordinates (X_1, X_2) and the time, then the motion is irrotational if*

$$\frac{\partial(\dot{x}_1, x_1)}{\partial(X_1, X_2)} + \frac{\partial(\dot{x}_2, x_2)}{\partial(X_1, X_2)} = 0$$

Ans. Let v_1, v_2 be the velocity components parallel to the axes of x_1 and x_2 respectively. Then we have

$$\dot{x}_1 = v_1,\ \dot{x}_2 = v_2$$

and
$$\frac{\partial v_1}{\partial X_1} = \frac{\partial v_1}{\partial x_1} \cdot \frac{\partial x_1}{\partial X_1} + \frac{\partial v_1}{\partial x_2} \cdot \frac{\partial x_2}{\partial X_1},$$

$$\frac{\partial v_1}{\partial X_2} = \frac{\partial v_1}{\partial x_1} \cdot \frac{\partial x_1}{\partial X_2} + \frac{\partial v_1}{\partial x_2} \cdot \frac{\partial x_2}{\partial X_2}$$

$$\frac{\partial v_2}{\partial X_1} = \frac{\partial v_2}{\partial x_1} \cdot \frac{\partial x_1}{\partial X_1} + \frac{\partial v_2}{\partial x_2} \cdot \frac{\partial x_2}{\partial X_1}$$

$$\frac{\partial v_2}{\partial X_2} = \frac{\partial v_2}{\partial x_1} \cdot \frac{\partial x_1}{\partial X_2} + \frac{\partial v_2}{\partial x_2} \cdot \frac{\partial x_2}{\partial X_2}.$$

Now, $$\frac{\partial(\dot{x}_1, x_1)}{\partial(X_1, X_2)} + \frac{\partial(\dot{x}_2, x_2)}{\partial(X_1, X_2)} = \frac{\partial(v_1, x_1)}{\partial(X_1, X_2)} + \frac{\partial(v_2, x_2)}{\partial(X_1, X_2)},$$

$$= \begin{vmatrix} \frac{\partial v_1}{\partial X_1} & \frac{\partial v_1}{\partial X_2} \\ \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \end{vmatrix} + \begin{vmatrix} \frac{\partial v_2}{\partial X_1} & \frac{\partial v_2}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{vmatrix} = \left(\frac{\partial v_1}{\partial X_1} \cdot \frac{\partial x_1}{\partial X_2} - \frac{\partial v_1}{\partial X_2} \cdot \frac{\partial x_1}{\partial X_1} \right) + \left(\frac{\partial v_2}{\partial X_1} \cdot \frac{\partial x_2}{\partial X_2} - \frac{\partial v_2}{\partial X_2} \cdot \frac{\partial x_2}{\partial X_1} \right)$$

$$=\left(\frac{\partial v_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial X_1}+\frac{\partial v_1}{\partial x_2}\cdot\frac{\partial x_2}{\partial X_1}\right)\frac{\partial x_1}{\partial X_2}-\left(\frac{\partial v_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial X_2}+\frac{\partial v_1}{\partial x_2}\cdot\frac{\partial x_2}{\partial X_2}\right)\cdot\frac{\partial x_1}{\partial X_1}$$

$$+\left(\frac{\partial v_2}{\partial x_1}\cdot\frac{\partial x_1}{\partial X_1}+\frac{\partial v_2}{\partial x_2}\cdot\frac{\partial x_2}{\partial X_1}\right)\cdot\frac{\partial x_2}{\partial X_2}-\left(\frac{\partial v_2}{\partial x_1}\cdot\frac{\partial x_1}{\partial X_2}+\frac{\partial v_2}{\partial x_2}\cdot\frac{\partial x_2}{\partial X_2}\right)\cdot\frac{\partial x_2}{\partial X_1}$$

$$=\frac{\partial v_1}{\partial x_2}\left(\frac{\partial x_1}{\partial X_2}\cdot\frac{\partial x_2}{\partial X_1}-\frac{\partial x_1}{\partial X_1}\cdot\frac{\partial x_2}{\partial X_2}\right)+\frac{\partial v_2}{\partial x_1}\left(\frac{\partial x_2}{\partial X_2}\cdot\frac{\partial x_1}{\partial X_1}-\frac{\partial x_2}{\partial X_1}\cdot\frac{\partial x_1}{\partial X_2}\right)$$

$$=\left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right)\frac{\partial(x_1,x_2)}{\partial(X_1,X_2)}$$

$$=0 \text{ iff } \frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}=0$$

i.e., *iff* motion is irrotational.

Ex.7. *Give examples of irrotational and rotational flows.*

Ans. Consider the fluid motion given by

$$v_1=kx_1, v_2=0, v_3=0, (k\neq 0)$$

Then $\vec{V}=kx_1\vec{i}$

So, $$\vec{\nabla}\times\vec{V}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}\\ v_1 & v_2 & v_3\end{vmatrix}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}\\ kx_1 & 0 & 0\end{vmatrix}=\vec{i}.0-\vec{j}.0+\vec{k}.0=\vec{0}$$

$\Rightarrow$ Fluid motion is irrotational.

Again consider another type of fluid motion which is given by $v_1=ax_2, v_2=0, v_3=0, (a\neq 0)$.

$\therefore \vec{\nabla}\times\vec{V}=-a\vec{k}\neq 0$

Hence the motion is rotational.

Ex.8. *The velocity components for a two dimensional fluid system can be given in the Eulerian system by* $v_1=2x_1+2x_2+3t, v_2=x_1+x_2+\frac{t}{2}$.

Find the displacement of a fluid particle in the Lagrangian system.

Ans. *Let* (X_1,X_2) *be the initial co-ordinates of the fluid particle which takes the position* (x_1,x_2) *after time* t.

Now $v_1 = \dfrac{dx_1}{dt} = 2x_1 + 2x_2 + 3t$

$$v_2 = \frac{dx_2}{dt} = x_1 + x_2 + \frac{t}{2}$$

which can be written as

$$\left(\frac{d}{dt} - 2\right)x_1 - 2x_2 = 3t \text{ i.e., } (D-2)x_1 - 2x_2 = 3t \text{ .... (i)}$$

and $$\left(\frac{d}{dt} - 1\right)x_2 - x_1 = t/2 \text{ i.e., } (D-1)x_2 - x_1 = t/2 \text{ .... (ii)}$$

where $D \equiv \dfrac{d}{dt}$

Operating (ii) by $D-2$, then we get

$$(D-2)(D-1)x_2 - (D-2)x_1 = \frac{1}{2}(D-2)t$$

or, $$\left(D^2 - 3D + 2\right)x_2 - 2x_2 - 3t = \frac{1}{2} - t \text{ [using (i)]}$$

or, $$\left(D^2 - 3D\right)x_2 = \frac{1}{2} + 2t$$

which is a 2nd order linear differential equation whose auxiliary equation is

$$m^2 - 3m = 0$$
$$\therefore m = 3, 0.$$

and C.F. is $c_1 + c_2 e^{3t}$

Now $P.I.$ $$= \frac{1}{D^2 - 3D}\cdot\left(\frac{1}{2} + 2t\right) = -\frac{1}{3D}\cdot\left(1 - \frac{D}{3}\right)^{-1}\left(\frac{1}{2} + 2t\right)$$

$$= -\frac{1}{3D}\left[1 + \frac{D}{3} + \frac{D^2}{9} + \ldots\ldots\right]\left(\frac{1}{2} + 2t\right)$$

$$= -\frac{1}{3D}\left[\frac{1}{2} + \frac{2}{3} + 2t\right]$$

$$= -\frac{1}{3D}\left[\frac{7}{6} + 2t\right]$$

$$= -\frac{1}{3}\left[\frac{7}{6}t + t^2\right]$$

Hence general solution of (ii) is

$$x_2 = c_1 + c_2 e^{3t} - \frac{1}{3}\left(\frac{7}{6}t + t^2\right) \quad \text{.....................(iv)}$$

and $$\frac{dx_2}{dt} = 3c_2 e^{3t} - \frac{1}{3}\left(\frac{7}{6} + 2t\right) \quad \text{.....................(v)}$$

Using (iv) and (v) in (ii) we get

$$x_1 = \frac{dx_2}{dt} - x_2 - \frac{t}{2} = -c_1 + 2c_2 e^{3t} - \frac{7}{18} - \frac{7}{9}t + \frac{t^2}{3} \quad \text{.....................(vi)}$$

Initially, $t = 0, x_1 = X_1, x_2 = X_2$ then from (iv) and (vi) we get

$$X_1 = -c_1 + 2c_2 - \frac{7}{18}$$

and $X_2 = c_1 + c_2$

Solving above two equations, we get

$$\left.\begin{aligned} c_1 &= \left(\frac{2X_2 - X_1}{3} - 7/54\right) \\ \text{and } c_2 &= \left(\frac{X_1 + X_2}{3} + 7/54\right) \end{aligned}\right\} \quad \text{...............(vii)}$$

Hence from (iv) and (vi), using (vii), we get

$$x_1 = \frac{1}{3}(X_1 - 2X_2) + \left[\frac{2}{3}(X_1 + X_2) + \frac{7}{27}\right]e^{3t} - \frac{7}{27} - \frac{7}{9}t + \frac{t^2}{3},$$

$$x_2 = \frac{1}{3}(2X_2 - X_1) + \left[\frac{(X_1 + X_2)}{3} + \frac{7}{54}\right]e^{3t} - \frac{7}{18}t - \frac{t^2}{3} - \frac{7}{54}.$$

which are the required displacement of the fluid particle in the Lagrangian system.

Ex.9. *Test whether the motion specified by* $\vec{V} = \frac{k^2(x_1\vec{j} - x_2\vec{i})}{x_1^2 + x_2^2}, (k = \text{constant})$ *is a possible motion for an* *incompressible fluid. If so, determine the equations of stream lines. Also let whether the motion is of the potential kind and if determine the velocity potential.*

Ans. Since $\vec{V} = \frac{k^2(x_1\vec{j} - x_2\vec{i})}{x_1^2 + x_2^2}$, therefore

$$v_1 = \frac{-k^2 x_2}{x_1^2 + x_2^2}, v_2 = \frac{k^2 x_1}{x_1^2 + x_2^2}, v_3 = 0$$

Now, $\frac{\partial v_1}{\partial x_1} = \frac{2k^2 x_1 x_2}{(x_1^2 + x_2^2)^2}, \frac{\partial v_2}{\partial x_2} = -\frac{2k^2 x_1 x_2}{(x_1^2 + x_2^2)^2}$

$$\therefore \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{2k^2 x_1 x_2}{(x_1^2 + x_2^2)^2} - \frac{2k^2 x_1 x_2}{(x_1^2 + x_2^2)^2} + 0 = 0$$

So, equation of continuity for incompressible fluid is satisfied and hence the given velocity is the possible fluid motion.

Stream lines are given by

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3}$$

or, $\frac{x_1^2 + x_2^2}{-k^2 x_2} dx_1 = \frac{x_1^2 + x_2^2}{k^2 x_2} dx_2 = \frac{dx_3}{0}$

$\Rightarrow$ from first two ratios, $\frac{dx_1}{-x_2} = \frac{dx_2}{x_1} \Rightarrow x_1^2 + x_2^2 = c_1$

and from 3rd ratio, $dx_3 = 0 \Rightarrow x_3 = \text{Constant} = c_2$.

Hence the stream lines are circle whose centres lie on x_3-axis.

Let φ be the velocity potential. Then

$$d\varphi = \frac{\partial \varphi}{\partial x_1} dx_1 + \frac{\partial \varphi}{\partial x_2} dx_2 + \frac{\partial \varphi}{\partial x_3} dx_3$$

$$= -v_1 dx_1 - v_2 dx_2 - v_3 dx_3$$

$$= -\left[\frac{-k^2 x_2}{x_1^2 + x_2^2} dx_1 + \frac{k^2 x_2}{x_1^2 + x_2^2} dx_2\right] (\because v_3 = 0)$$

$$= k^2\left[\frac{x_2 dx_1}{x_1^2+x_2^2} - \frac{x_1 dx_2}{x_1^2+x_2^2}\right] = k^2\left(Mdx_1 + Ndx_2\right), \text{ say}$$

where $M = \frac{x_2}{x_1^2+x_2^2}$ and $N = \frac{-x_1}{x_1^2+x_2^2}$

Now $\frac{\partial M}{\partial x_2} = \frac{1}{x_1^2+x_2^2} + \frac{x_2(-2x_2)}{\left(x_1^2+x_2^2\right)^2} = \frac{x_1^2-x_2^2}{\left(x_1^2+x_2^2\right)^2}$,

$$\frac{\partial N}{\partial x_1} = -\left[\frac{1}{x_1^2+x_2^2} + \frac{x_1(-2x_1)}{\left(x_1^2+x_2^2\right)^2}\right] = \frac{x_1^2-x_2^2}{\left(x_1^2+x_2^2\right)^2}$$

i.e., $\frac{\partial M}{\partial x_2} = \frac{\partial N}{\partial x_1}$

i.e., $Mdx_1 + Ndx_2$ is an exact. So, the solution of it is given by

$$\varphi = \int \frac{k^2 x_1 dx_1}{x_1^2+x_2^2} + c_3$$

$$= k^2 \tan^{-1}\left(\frac{x_1}{x_2}\right) + c_3$$

Hence the velocity potential is $\varphi = k^2 \tan^{-1}\left(\frac{x_1}{x_2}\right)$, omiting constant.

Ex.10. *Show that the variable ellipsoid* $\frac{x^2}{a^2x^2t^4} + kt^2\left[\frac{y^2}{b^2}+\frac{z^2}{c^2}\right] = 1$ *is a possible form for the boundary surface of a liquid at any time t.*

Ans. Let $F(x,y,z,t) = \frac{x^2}{a^2k^2t^4} + kt^2\left[\frac{y^2}{b^2}+\frac{z^2}{c^2}\right] - 1 = 0$ .... (i)

Now, the surface $F = 0$ to be a possible form of boundary surface, if

$$\frac{dF}{dt} = \vec{V}\cdot\vec{\nabla}F + \frac{\partial F}{\partial t} = 0$$

i.e. $u\frac{dF}{dx} + v\frac{\partial F}{\partial y} + w\frac{dF}{dz} + \frac{\partial F}{\partial t} = 0$ .... (ii) i.e., if $\frac{u2x}{a^2k^2t^4}+vkt^2\frac{2y}{b^2}+wkt^2\cdot\frac{2z}{c^2}-\frac{4x^2}{a^2k^2t^5}+2kt\left[\frac{y^2}{b^2}+\frac{z^2}{c^2}\right]=0$

i.e., if $\frac{2x}{a^2k^2t^4}\left(u-\frac{2x}{t}\right)+\frac{2k}{b^2}t^2y\left(v+\frac{y}{t}\right)+\frac{2k}{c^2}t^2z\left(w+\frac{z}{t}\right)=0$

Hence (ii) is satisfied if we take

$$u-\frac{2x}{t}=0, v+\frac{y}{t}=0, w+\frac{z}{t}=0$$

i.e., if $u=\frac{2x}{t}, v=-y/t, w=-z/t$

It will be a justified step if the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

is satisfied.

Now, $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{2}{t}-\frac{1}{t}-\frac{1}{t}=0$

Hence (i) is a possible form of boundary surface.

Ex.11. *Show that* $\frac{x^2}{a^2}\tan^2 t+\frac{y^2}{b^2}\cot^2 t-1=0$ *is a possible form of boundary surface and find an expression for normal velocity.*

Ans. Let $F=\frac{x^2}{a^2}\tan^2 t+\frac{y^2}{b^2}\cot^2 t-1=0$ .... (i)

Now, the surface $F=0$ to be a possible form of boundary surface if

$$\frac{dF}{dt}=0 \text{ i.e., } \frac{\partial F}{\partial t}+\vec{V}\cdot\vec{\nabla}F=0$$

i.e., if $u\frac{dF}{dx}+v\frac{\partial F}{\partial y}+w\frac{dF}{dz}+\frac{\partial F}{\partial t}=0$

i.e., if $\frac{2x}{a^2}\tan^2 t\left(u+\frac{x\sec^2 t}{\tan t}\right)+\frac{2y}{b^2}\cot^2 t\left(v-\frac{y\,\text{cosec}^2 t}{\cot t}\right)=0$ .... (ii)

Thus (ii) will be satisfied if we take

$$u+\frac{x\sec^2 t}{\tan t}=0 \text{ and } v-y\frac{\operatorname{cosec}^2 t}{\cot t}=0$$

i.e., if $u=-\frac{x}{\sin t\cos t}, v=\frac{y}{\sin t\cos t}$.

This will be a justifiable step if the equation of continuity namely $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$ is satisfied.

Now $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=-\frac{1}{\sin t\cos t}+\frac{1}{\sin t\cos t}+0=0$

Hence the given surface is a possible form of boundary surface.

$$\text{The normal velocity}=\frac{-\frac{\partial F}{\partial t}}{|\vec{\nabla} F|}=-\frac{\frac{\partial F}{\partial t}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2}}$$

$$=-\frac{\left(\frac{2x^2}{a^2}\tan t\sec^2 t-\frac{2y^2}{b^2}\cot t Co\sec^2 t\right)}{\sqrt{\left(\frac{2x}{a^2}\tan t\right)^2+\left(\frac{2y}{b^2}\cot^2 t\right)^2}}$$

$$=-\frac{b^2x^2\tan t\sec^2 t-a^2y^2\cot t Co\sec^2 t}{\left(b^4x^2\tan^4 t+a^4y^2\cot^4 t\right)^{1/2}}$$

Ex.12. *Show that all necessary and sufficient condition can be satisfied by a velocity potential of the form $\varphi=\alpha x^2+\beta y^2+\gamma z^2$ and the bounding surface of the form $F=ax^4+by^4+cz^4-X(t)=0$, where $X(t)$ is a given function of time and $\alpha,\beta,\gamma,a,b,c$ suitable functions of the time.*

Ans. Let $F=ax^4+by^4+cz^4-X(t)=0$ .... (i)

and $\varphi=\alpha x^2+\beta y^2+\gamma z^2$ .... (ii)

To prove that φ satisfies all the necessary condition i.e., equation of continuity

i.e., $\nabla^2\varphi=0$

$$\text{or,}\quad \frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0$$

$$\text{or,}\quad 2\alpha+2\beta+2\gamma=0$$

$$\text{or,}\quad \alpha+\beta+\gamma=0\ \ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

The velocity potential φ has to satisfy this condition.

To prove $F = 0$ satisfies the condition of boundary surface.

Since $u=-\frac{\partial\varphi}{\partial x}, v=-\frac{\partial\varphi}{\partial y}, w=-\frac{\partial\varphi}{\partial z}$.

Therefore, $u=-2\alpha x, v=-2\beta y, w=-2\gamma z$.

For the condition of boundary surface we have $\frac{dF}{dt}=0$.

$$\text{i.e.,}\quad u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}+\frac{\partial F}{\partial t}=0$$

$$\text{i.e.,}\quad -2\alpha x\cdot 4ax^3-2\beta y\cdot 4by^3-2\gamma x\cdot 4cz^3+x^4a'+y^4b'+z^4c'-X'(t)=0$$

$$\text{i.e.,}\quad x^4(a'-8a\alpha)+y^4(b'-8b\beta)+z^4(c'-8c\gamma)-X'(t)=0\ \ldots\ldots\ldots\text{(iv)}$$

Since (i) and (iv) both have to hold for all points (x, y, z) on the surface and hence they should be identical.

Compairing, we get

$$\frac{a'-8a\alpha}{a}=\frac{b'-8b\beta}{b}=\frac{c'-8c\gamma}{c}=\frac{X'(t)}{X(t)}$$

From 1st and 4th ratios

$$\frac{da}{dt}-8a\alpha=\frac{a}{X}\cdot\frac{dX}{dt}$$

$$\text{or,}\quad \frac{da}{a}=8\alpha dt+\frac{dX}{dt}$$

$$\text{Int.,}\quad \log_e a=\log_e X+8\int\alpha dt$$

$$\text{Similarly,}\quad \log_e b=\log_e X+8\int\beta dt$$

$$\text{and}\quad \log_e c=\log_e X+8\int\gamma dt$$

such that $\alpha+\beta+\gamma=0$

The surface $F=0$ will have to satisfy these conditions for the possible form of boundary surface.

Ex.13. *Determine the restriction on* f_1, f_2, f_3 if $\frac{x^2}{a^2}f_1(t)+\frac{y^2}{b^2}f_2(t)+\frac{z^2}{c^2}f_3(t)=1$ *is a possible form of boundary surface of a liquid.*

Ans. Let $F=\frac{x^2}{a^2}f_1(t)+\frac{y^2}{b^2}f_2(t)+\frac{z^2}{c^2}f_3(t)-1=0$ .... (i)

For boundary surface F must satisfies the equation

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0$$

i.e., $$\frac{x^2}{a^2}f_1'+\frac{y^2}{b^2}f_2'+\frac{z^2}{c^2}f_3'+u\frac{2x}{a^2}f_1+v\frac{2y}{b^2}f_2+w\frac{2y}{c^2}f_3=0$$

i.e., $$\frac{2x}{a^2}f_1\left(u+\frac{xf_1'}{f_1}\right)+\frac{2y}{b^2}f_2\left(v+\frac{yf_2'}{f_2}\right)+\frac{2z}{c^2}f_3\left(w+\frac{2f_3'}{f_3}\right)=0$$

If we take, $u+\frac{xf_1'}{f_1}=0, v+\frac{yf_2'}{f_2}=0, w+\frac{zf_3'}{f_3}=0$ then above equation is satisfied. Now this will be justifiable step if the values u, v, w satisfy the equation of continuity, i.e.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

So, we have $$-\left[\frac{f_1'}{f_1}+\frac{f_2'}{f_2}+\frac{f_3'}{f_3}\right]=0$$

i.e., $$\frac{df_1}{f_1}+\frac{df_2}{f_2}+\frac{df_3}{f_3}=0$$

Int., $\log_e f_1(t)f_2(t)f_3(t)=\log_e c$, c being constant.

i.e., $f_1(t)\cdot f_2(t)\cdot f_3(t)=c$.

5.26 Self Assessment Questions:

1. Show that the following motions are possible for an incompressible fluid:

i) $v_1=cx_1, v_2=cx_2, w=-2cx_3$;

ii) $v_1 = \frac{cx_1}{r^3}, v_2 = \frac{cx_2}{r^3}, w = cz$

where $r^2 = x_1^2 + x_2^2 + x_3^2$ and c is constant.

2. Find the stream lines when $v_1 = ax_1, v_2 = -ax_2, w = c$, a and c are constants.

3. The velocity field at a point is given by $\vec{V} = \left(\frac{x_1}{t}, x_2, 0\right)$. Find the path lines and stream lines.

4. For a two dimensional flow the velocities at a point in a fluid may be expressed in the Eulerian co-ordinates by $v_1 = x_1 + x_2 + 2t$ and $v_2 = 2x_2 + t$. Determine the Lagrange co-ordinates as functions of initial positions X_1 and X_2 and the time t.

5. If V is the resultant velocity at any point of a fluid which is moving irrotationally in two dimensions; prove that

$$\left(\frac{\partial V}{\partial x_1}\right)^2 + \left(\frac{\partial V}{\partial x_2}\right)^2 = V\nabla^2 V.$$

6. Prove that a surface of the form $ax_1^4 + bx_2^4 + cx_3^4 - X(t) = 0$ is a possible form of boundary surface of a homogeneous liquid at time t, the velocity potential of the liquid motion being $\varphi = (\beta - \gamma)x_1^2 + (\gamma - \alpha)x_2^2 + (\alpha - \beta)x_3^2$ where X, α, β, γ are given functions of time t and a, b, c are suitable functions of time.

7. Show that the ellipsoid $\frac{x_1^2}{a^2k^2t^{2n}} + \left(\frac{x_2^2}{b^2} + \frac{x_3^2}{c^2}\right)kt^n = 1$ is a possible form of boundary surface.

8. In the steady motion of homogeneous liquid if the surfaces $f_1 = a_1, f_2 = a_2$ define the stream lines, prove that the most general values of the velocity components v_1, v_2, v_3 are

$$F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(x_2, x_3)}, F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(x_3, x_1)}, F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)}.$$

9. Show that $\frac{x_1^2}{a^2}\varphi(t) + \frac{x_2^2}{b^2} \cdot \frac{1}{\varphi(t)} = 1$ is a possible form of boundary surface of a liquid.

10. Show that $\varphi = (x_1 - t)(x_2 - t)$ represents the possible velocity potential of an incompressible fluid. Find the stream lines and path lines.

11. Determine the constants λ, μ, γ in order that liquid motion is possible when velocity components at (x_1, x_2, x_3) are

$$v_1 = \frac{x_1 + \lambda r}{r(x_1 + r)}, v_2 = \frac{x_2 + \mu r}{r(x_1 + r)}, v_3 = \frac{x_3 + \gamma r}{r(x_1 + r)} \text{ where } r^2 = x_1^2 + x_2^2 + x_3^2.$$

5.27 Further Suggested Readings:

1. Continuum Mechanics: T.J. Chung, Prentice-Hall.
2. Schaum's Outline of Theory and Problem of Continuum Mechanics: Gedrge R. Mase, McGraw-Hill.
3. Continuum Mechanics: A.J.M. Spencer, Longman.
4. Mathematical Theory of Continuum Mechanics: R.N. Chatterjee, Narosa Publishing House.
5. Foundation of Fluid Mechanics: S.W. Yuan, Prentice-Hall.
6. Fluid Dynamics: J.K. Goyal, K.P. Gupta, Pragati Prakashan.
7. Textbook of Fluid Dynamics: F. Chorlton, CBS Publishers and Distributors.
8. Theory of Elasticity : Yu. Amenzade, Mir Publishers, Moscow.
9. Applied Elasticity : C.T. Wang, McGraw-Hill.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-V **Group – A** **Marks – 50**

Module No. - 54

Mechanics of Continous Media

STRUCTURE :

6.1 Introduction :

This the continuation of the Module No. 53. Here we discuss remaining portion of the perfect fluid. To understand about the motion of perfect fluid this part is essential.

6.2 Objectives :

In this module, the students will learn the technique and methods for vorticity vector, impulsive motion, flow, circulation, energy theory, two-dimensional motion, stream function, complex potential, sources, sinks, doublet, image, circle-theorem etc.

6.3 Key Words :

Vorticity vector, vortex line, vortex tube, Beltranic vector, Impulsive motion, equations for impulsive action, flow, circulation, Kelvin's circulation theorem, energy equation, minimum energy theorem, uniqueness theorem, motion in 2D, stream function, complex potential, Cauchy-Riemann equation, sources, sinks, strength, doublet, strength of doublet, circle theorem.

6.4 The Vorticity Vector :

Now we consider the flows for which curl $\vec{V}$ is non-zero i.e., $\vec{\nabla}\times\vec{V}\neq\vec{0}$. Then the vector

$$\vec{W}=\vec{\nabla}\times\vec{V}$$

is called the *vorticity vector*.

The mathematician Lamb, Milne-Thompson, Rutherford, Goldstein etc. follow the definition $\vec{W} = \vec{\nabla} \times \vec{V}$

where as Birkhoff, Robertson etc. follow the defintion $\vec{W} = \frac{1}{2}\vec{\nabla} \times \vec{V}$.

Let $\vec{W} = \vec{W}(\xi,\eta,\zeta) = \xi\vec{i} + \eta\vec{j} + \zeta\vec{k}$ then

$$\xi\vec{i} + \eta\vec{j} + \zeta\vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \quad \text{where } \vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\text{which} \Rightarrow \xi = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right),\ \eta = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right),\ \zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

It is clear that non-zero verticity vector exist for rotational fluid motion and for irrotational fluid motion must have $\vec{\nabla} \times \vec{V} = \vec{0}$ i.e. $\vec{W} = \vec{0}$. Hence one can defined a fluid motion is irrotational if $\vec{W} = \vec{0}$ i.e. $\xi = 0,\ \eta = 0,\ \zeta = 0$ otherwise rotational.

6.5 Vortex line:

Vortex line is a curve such that tangent at each point of it at any instant of time is in the direction of vorticity vector at that point. It means that $\vec{W}$ is parallel to $d\vec{r}$. Hence

$$\vec{W} \times d\vec{r} = \vec{0}$$

$$\Rightarrow \frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}$$

$$\text{i.e., } \frac{dx}{\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}} = \frac{dy}{\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}} = \frac{dz}{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}},$$

which are the differential equations of a vortex lines. In general vortex lines do not coincide with the stream lines.

6.6 Vortex tube:

The vortex lines drawn through each point of a closed curve enclose a tubular space in the fluid is vortex tube.

A vortex tube of infinitesimal cross section is called a *vortex filament* or *vortex* simply.

The value of $|\vec{n}.\vec{W}|dS$ is called the *strength* of the vortex tube. A vortex tube whose strength is unity is called a unit vortex tube. Since

$$\int_S \vec{n}.\vec{W}dS = \int_V \vec{\nabla}.\vec{W}dV = 0$$

so, the total strength of vortex tubes emerging from S is equal to that entering S. This means that vortex lines and tubes can not originate or terminate at internal points in a fluid. They can only form closed curves or terminate on boundaries.

In case of *smoke rings*, the vortex lines form closed curves. On the other hand, the vortex lines in a *whirlpool terminate on the boundary of the fluid.*

If the fluid motion $\vec{V}$ is parallel to $\vec{W}$ then the fluid motion is said to be *Beltranic flow*. In this case $\vec{V}\times\vec{W}=\vec{0}$ and $\vec{V}$ is called *Beltranic Vector*.

Ex.1. *The velocity vector in the flow field is given by* $\vec{V}=\vec{i}\left(Az-By\right)+\vec{j}\left(Bx-Cz\right)+\vec{k}\left(Cy-Ax\right)$ *where A, B, C are non-zero constants. Determine the equations of the vortex lines.*

Ans. Let $\vec{W}=\xi\vec{i}+\eta\vec{j}+\zeta\vec{k}$ be the vorticity vector $=\vec{\nabla}\times\vec{V}$.

$$\therefore\ \vec{W}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ Az-By & Bx-Cz & Cy-Ax\end{vmatrix}=2C\vec{i}+2A\vec{j}+2B\vec{k}$$

$$\Rightarrow \xi=2C,\ \eta=2A,\ \zeta=2B$$

Hence vortex lines are given by

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dz}{\zeta}$$

i.e. $$\frac{dx}{2C}=\frac{dy}{2A}=\frac{dz}{2B}$$

$$\frac{dx}{C}=\frac{dy}{A}=\frac{dz}{B}$$

From 1st and 2nd ratios, we have

$$Adx = Cdy \Rightarrow Ax - Cy = C_1 \ldots \text{(i)}$$

From 2nd and 3rd ratios, we have

$$Bdy = Adz \Rightarrow By - Az = C_2 \ldots \text{(ii)}$$

Hence the vortex lines are given by the equations (i) and (ii) together

Ex.2. *If velocity distribution is $\vec{V} = \vec{i}\left(Ax^2yt\right) + \vec{j}\left(By^2zt\right) + \vec{k}\left(Czt^2\right)$ where A, B, C are constants, then find acceleration and vorticity components.*

Ans. Let $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$. Then $u = Ax^2yt,\ v = By^2zt,\ w = Czt^2$.

If a_1, a_2, a_3 be the components of the acceleration $\vec{a}$. Then

$$a_1 = \frac{du}{dt} = \frac{\partial u}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u$$

$$= Ax^2y + \left(Ax^2yt\right)\left(2Axyt\right) + \left(By^2zt\right)\left(Ax^2t\right) + \left(Czt^2\right).0$$

$$= Ax^2y\left[1 + 2Axyt^2 + Byzt^2\right]$$

$$a_2 = \frac{dv}{dt} = \frac{\partial v}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)v$$

$$= By^2z + \left(Ax^2yt\right)(0) + \left(By^2zt\right)\left(2Byzt\right) + \left(Czt^2\right)\left(By^2t\right)$$

$$= By^2z\left[1 + 2Byzt^2 + Ct^3\right]$$

$$a_3 = \frac{dw}{dt} = \frac{\partial w}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)W$$

$$= 2Czt + \left(Ax^2yt\right)(0) + \left(By^2zt\right)(0) + \left(Czt^2\right)\left(Ct^2\right)$$

$$= Czt\left[2 + Ct^3\right]$$

Let ξ, η, ζ be the components of the vorticity vector $\vec{W}$. Then

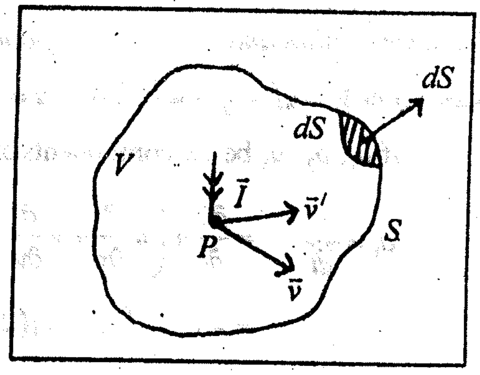
$$\vec{W} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ax^2yt & By^2zt & Czt^2 \end{vmatrix} = -By^2t\vec{i} - Ax^2t\,\vec{k}$$

$$\therefore \xi = -By^2t,\ \eta = 0,\ \zeta = -Ax^2t.$$

6.7 Impulsive Motion:

Suppose sudden velocity changes are produced at the boundaries of an incompressible fluid or that impulsive forces are applied to its interior. Then the disturbances produced in both cases are propagated instantaneously throughout the fluid.

6.8 Equations for impulsive Action:



Let us consider an arbitrary closed surface S moving with a non-viscous fluid such that it enclose a volume V. Let $\vec{V}$ and $\vec{V}'$ be the fluid velocity at P within S just before the impulse and just after the impulse. Let ρ be the fluid density at P. Suppose $\vec{I}$ is the external impuse per unit mass and $\vec{w}$ the impulsive pressure on a surface element dS. Also let $\vec{n}$ be the unit outward normal vector. Now, impulse on the volume element dV is $\vec{I}\rho dV$ and that on dS is $-\vec{n}\vec{w}dS$. Hence total impulse applied to the fluid particle is

$$\int_V \vec{I}\rho dV + \int_S (-\vec{n}\vec{w}dS) = \int_V \vec{I}\rho dV - \int_V \vec{\nabla}\vec{w}dV = \int_V (\rho\vec{I} - \vec{\nabla}\vec{w})dV$$

(by Gauss theorem)

The total change of momentum is $\int_V \rho(\vec{V}' - \vec{V})dV$.

Hence from, change of momentum = total impulsive force, we get

$$\int_V \rho(\vec{V}' - \vec{V})dV = \int_V (\rho\vec{I} - \vec{\nabla}\vec{w})dV$$

or, $$\int_V \left[\rho(\vec{V}' - \vec{V}) - \rho\vec{I} + \vec{\nabla}\vec{w}\right]dV = 0$$

Since V is an arbitrarily small volume, our usual assumption of fluid continuity yields the result

$$\rho(\vec{V}' - \vec{V} - \rho\vec{I} + \vec{\nabla}\vec{w}) = 0$$

or, $$\vec{V}' - \vec{V} = \vec{I} - \frac{1}{\rho}\vec{\nabla}\vec{w} \quad \text{... (1)}$$

at each point P of the fluid. This is the required equation for impulsive action.

If $\vec{I} = (I_x, I_y, I_z)$, $\vec{V}' = (u, v, w)$, $\vec{V} = (u_0, v_0, w_0)$ then the cartesian equivalent of (1) is

$$\left.\begin{aligned} u - u_0 &= I_x - \frac{1}{\rho}\frac{\partial \tilde{w}}{\partial x} \\ v - v_0 &= I_y - \frac{1}{\rho}\frac{\partial \tilde{w}}{\partial y} \\ \text{and } w - w_0 &= I_z - \frac{1}{\rho}\frac{\partial \tilde{w}}{\partial z} \end{aligned}\right\} \quad \ldots\ldots(2)$$

Deduction (I) *Vorticity in a non-viscous incompressible fluid is never generated by impulses if the external forces are conservative.*

Proof.: External impulses are conservative implies that

$$\vec{I} = -\vec{\nabla}\Omega$$

and for incompressible we have $\rho =$ constant. So, from (1), we get

$$\vec{V}' - \vec{V} = -\vec{\nabla}\left(\Omega + \frac{\tilde{w}}{\rho}\right)$$

or, $\vec{\nabla} \times \left(\vec{V}' - \vec{V}\right) = \vec{0}$ ($\because$ curl (grad.) = 0)

or, $\vec{\nabla} \times \vec{V}' = \vec{\nabla} \times \vec{V}$

or, $\vec{W}' = \vec{W}$

Hence the result.

II. *Prove that* $\nabla^2 \tilde{w} = 0$ *under suitable conditions.*

Proof. Let the external impulse be absent so that $\vec{I} = \vec{0}$.

Also, let ρ be constant. Then from (1), we get

$$\vec{V}' - \vec{V} = -\vec{\nabla}\left(\frac{\tilde{w}}{\rho}\right)$$

or, $\vec{\nabla} \cdot \left(\vec{V}' - \vec{V}\right) = -\vec{\nabla} \cdot \left(\vec{\nabla}\left(\frac{\tilde{w}}{\rho}\right)\right) = -\nabla^2\left(\frac{\tilde{w}}{\rho}\right)$

or, $\nabla^2 \tilde{w} = \rho\left[-\vec{\nabla} \cdot \vec{V}' + \vec{\nabla} \cdot \vec{V}\right] = \rho(-0 + 0) = 0$

i.e. $\nabla^2 \tilde{w} = 0$

When $\vec{\nabla}\cdot\vec{V}'=0=\vec{\nabla}\cdot\vec{V}$ i.e. when the fluid is incompressible.

Note: If the motion is irrotational, then

$$\vec{V}=-\vec{\nabla}\varphi,\ \vec{V}'=-\vec{\nabla}\varphi'$$

So, from (1) we get

$$-\vec{\nabla}\varphi'+\vec{\nabla}\varphi=-\vec{\nabla}\left(\frac{\tilde{w}}{\rho}\right)\ \textit{for}\ \vec{I}=\vec{0}.$$

or, $\vec{\nabla}\left[\rho(\varphi'-\varphi)-\tilde{w}\right]=\vec{0}$

or, $\rho(\varphi'-\varphi)=\tilde{w}$, neglecting constant of integration.

III. *Prove that $\tilde{w}=\rho\varphi$ under suitable conditions.*

Proof. Let us assume that $\vec{I}=\vec{0}$ and ρ = constant. Then from (1)

We get $\vec{V}'-\vec{0}=\vec{0}-\dfrac{1}{\rho}\vec{\nabla}\tilde{w}$

if the motion starts from rest.

Since the motion starts from rest by the application of impulsive pressure and hence it must be irrotational.

Then $\vec{V}'=-\vec{\nabla}\varphi$, say.

$$\therefore\quad -\vec{\nabla}\varphi=-\frac{1}{\rho}\vec{\nabla}\tilde{w}$$

or, $\vec{\nabla}\left(\varphi-\dfrac{\tilde{w}}{\rho}\right)=\vec{0}$

$$\Rightarrow \varphi-\frac{\tilde{w}}{\rho}=\text{constant}$$

or, $\rho\varphi=\tilde{w}$, neglecting constant.

Note: If $\vec{I}=\vec{0}$ and $\vec{V}'=\vec{0}$ then from (1),

$$-\vec{V}=-\frac{1}{\rho}\vec{\nabla}\tilde{w}$$

and if $\vec{V}=(u,0,0)$ then we have

$$\frac{d\tilde{w}}{dx}=\rho u$$

6.6 Flow :

Let us consider any two points A and B in a fluid.

The value of the integral

$$\int_A^B \vec{V}\cdot d\vec{r} = \int_A^B (udx + vdy + wdz)$$

taken along any path in the fluid, is called *flow from A to B along that path.*

If the motion is irrotational, then the flow is

$$\int_A^B \vec{V}\cdot d\vec{r} = \int_A^B (-\vec{\nabla}\varphi)\cdot d\vec{r} \quad (\because \vec{V} = -\vec{\nabla}\varphi \text{ for irrotational motion})$$

$$= \int_A^B (-d\varphi) = \varphi_A - \varphi_B$$

where φ_B and φ_A denote velocity potentials at B and A respectively.

6.10 Circulation :

Flow along a closed path C is defined as circulation and it denoted by Γ. So,

$$\Gamma = \oint_C (udx + vdy + wdz) = \oint_C \vec{V}\cdot d\vec{r}$$

If the motion is irrotational, then $\vec{V} = -\vec{\nabla}.\varphi$ and hence

$$\Gamma = \oint_C (-\vec{\nabla}\varphi)\cdot d\vec{r} = \oint_C (-d\varphi) = \varphi_A - \varphi_A = 0$$

(since the points A and B are coincide for a closed path).

6.11 Kelvin's Circulation Theorem :

For an inviscid fluid the circulation around any closed circuit of fluid particles, moving along with the fluid, remains constant provided that the body forces are conservative and the pressure is a single-valued function of density only.

Proof. Euler's equation of motion, for inviscid fluid, is

$$\frac{d\vec{V}}{dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p$$

$$= -\vec{\nabla}\Omega - \frac{1}{\rho}\vec{\nabla}p \quad \text{... (i)}$$

where $\vec{F} = -\vec{\nabla}\Omega$, Ω is the potential of the conservative body forces.

Let C be a closed path (circuit) of fluid particles moving along with the fluid and suppose that $\vec{r}$, $\vec{r}+d\vec{r}$ are the position vector of neighbouring particles P, P' of C referred to a fixed origin O. Then the circulation round C is

$$\Gamma = \oint_C \vec{\nabla} \cdot d\vec{r}$$

Now, $$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint_C \vec{V} \cdot d\vec{r} \quad = \oint_C \left\{ \frac{d\vec{V}}{dt} \cdot d\vec{r} + \vec{V} \cdot \frac{d}{dt}(d\vec{r}) \right\}$$

$$= \oint_C \left\{ \frac{d\vec{V}}{dt} \cdot d\vec{r} + \vec{V} \cdot d\vec{V} \right\} \left[\because \frac{d}{dt}(d\vec{r}) = d\left(\frac{d\vec{r}}{dt}\right) \right]$$

$$= \oint_C \left\{ \left(-\vec{\nabla}\Omega - \frac{1}{\rho}\vec{\nabla}p \right) \cdot d\vec{r} + \vec{V} \cdot d\vec{V} \right\} \text{ [by (i)]}$$

$$= \oint_C \left\{ \left(-\vec{\nabla}\Omega - \frac{1}{\rho}\vec{\nabla}p \right) \cdot d\vec{r} + d\left(\frac{1}{2}V^2\right) \right\}$$

$$= \oint_C \left\{ \left(-d\Omega - \frac{dp}{\rho} \right) + d\left(\frac{1}{2}V^2\right) \right\} \left(\because \vec{\nabla} \cdot d\vec{r} \equiv d \right)$$

$$= \oint_C d\left(-\Omega + \frac{1}{2}V^2 \right) - \oint_C \frac{dp}{\rho}$$

$$= \left[-\Omega + \frac{1}{2}V^2 - \int \frac{dp}{\rho} \right]_C$$

$$= 0$$

(since R.H.S. involved by single-valued and on passing once round the circuit).

So, $\frac{d\Gamma}{dt} = 0$

$\Rightarrow \Gamma$ = constant, in the moving circuit.

i.e., circulation is constant along C for all times.

Corollary - 1. In a closed circuit C of fluid particles moving under the same conditions as before, $\int_S \vec{n}\cdot(\vec{\nabla}\times\vec{V})\,dS = \int_S \vec{n}\cdot\vec{W}dS$ is constant, where S is an open surface with C as rim.

Proof. The circulation along the closed path C, for all time t, is $\Gamma = \oint_C \vec{V}\cdot d\vec{r} = \int_S \vec{n}\cdot(\vec{\nabla}\times\vec{V})\,dS$, (by Stoke's theorem) which implies that $\int_S \vec{n}\cdot(\vec{\nabla}\times\vec{V})\,dS = \int_S \vec{n}\cdot\vec{W}dS$ = constant as Γ is constant (under the conditions of Kelvin's theorem).

Corollary - 2. Under the above conditions, if any portion of the moving fluid once becomes irrotational, then it remains so for all subsequent times.

Proof. Let us suppose that at some instant the fluid on S becomes irrotational. Then $\vec{V} = -\vec{\nabla}\varphi$ and hence the vorticity vector $\vec{W} = \vec{\nabla}\times\vec{V} = -\vec{\nabla}(\vec{\nabla}\varphi) = \vec{0}$ at all points of S and the last result shows that

$$\Gamma = \oint_C \vec{V}\cdot d\vec{r} = \int_S \vec{n}\cdot\vec{W}dS = 0 \text{ for } \vec{W} = \vec{0}$$

at the given instant and hence at all subsequent times.

Thus at any later stage,

$$\int_S \vec{n}\cdot\vec{W}dS = 0$$

If we now take S to be a non-zero infinitesimal element, say dS, then to the first order $\vec{n}\cdot\vec{W}dS = 0$ showing that in general $\vec{W} = \vec{0}$ at any point of dS, i.e., that the motion stays irrotational.

Corollary - 3. With the above conditions, the vortex lines moves along with the fluid.

Proof. For a surface S moving with the fluid we have seen that $\int_S \vec{n}\cdot\vec{W}dS$ = constant. This integral represents total strength of vortex tubes passing through S. Which shows that the vortex tubes move with the fluid. By taking S vanishingly small, we infer that the vortex lines move with the fluid. .

6.2 Energy Equation :

The time rate of change of total energy (i.e. sum of kinetic, potential, intrinsic energies) of any portion of the fluid is equal to the rate of work done by external pressure across its boundary provided the external forces are conservative, i.e.

$$\frac{d}{dt}(T+W+I)=\iint p(\vec{n}\cdot\vec{v})dS,$$

where *T, W, I* are Kinetic energy, Potential energy, Intrinsic energy respectively.

Proof. Let us consider an arbitrary closed surface *S* moving with a non-viscous fluid such that it encloses a volume *V*. Let $\vec{n}$ be the unit inward drawn normal vector on an element *dS*. Let the force be conservative so that $\vec{F}=-\vec{\nabla}\Omega$, where Ω is time-independent scalar potential function.

So, $\frac{\partial\Omega}{\partial t}=0$ and hence

$$\frac{d\Omega}{dt}=\frac{\partial\Omega}{\partial t}+(\vec{V}\cdot\vec{\nabla})\Omega=(\vec{V}\cdot\vec{\nabla})\Omega \text{ ...(i)}$$

Let *T, W* and *I* denotes the kinetic energy, potential energy and intrinsic energy respectively. Since Ω is force potential per unit mass and hence

$$W=\int\Omega\,dm=\int_V\Omega\,\rho dV$$

$$T=\int_V\frac{1}{2}\rho\left|\vec{V}\right|^2dV=\int_V\frac{1}{2}\left|\vec{V}\right|^2\rho dV$$

Since elementary mass remains invariant through the motion and so $\frac{d}{dt}(\rho dV)=0$.

Now, $$\frac{dT}{dt}=\frac{1}{2}\int_V\frac{d\left|\vec{V}\right|^2}{dt}\cdot\rho dV+\frac{1}{2}\int_V\left|\vec{V}\right|^2\frac{d(\rho dV)}{dt}$$

$$=\frac{1}{2}\int_V\vec{V}\cdot\frac{d\vec{V}}{dt}\rho dV+0\left(\because\left|\vec{V}\right|^2=\vec{V}\cdot\vec{V}\text{ and }\frac{d}{dt}(\rho dV)=0\right)$$

and $$\frac{dW}{dt}=\int_V\frac{d\Omega}{dt}\rho dV+\int_V\Omega\frac{d(\rho dV)}{dt}=\int_V\frac{d\Omega}{dt}\rho dV$$

Intrinsic energy *E* per unit mass of the fluid is defined as the work done by the unit mass of the fluid against external pressure *p* under the supposed relation between pressure and density ρ from its actual state to some standard state in which pressure and density are p_0 and ρ_0 respectively. Then

$$I=\int_V E\rho dV$$

where $E = \int_V^{V_0} p\,dV = \int_\rho^{\rho_0} p\,d\left(\frac{1}{\rho}\right) \left(\because \rho = \frac{1}{V}\right)$

$$= -\int_\rho^{\rho_0} p\frac{d\rho}{\rho^2} = \int_{\rho_0}^{\rho} \frac{p}{\rho^2}\,d\rho$$

$$\Rightarrow \frac{dE}{d\rho} = \frac{p}{\rho^2} \quad \text{... (ii)}$$

$$\therefore \frac{dI}{dt} = \int_V \left[\frac{dE}{dt}\rho dV + E\frac{d}{dt}(\rho dV)\right] = \int_V \frac{dE}{dt}\rho dV$$

$$= \int_V \frac{dE}{d\rho}\cdot\frac{d\rho}{dt}\rho dV = \int_V \frac{p}{\rho^2}\frac{d\rho}{dt}\rho dV \quad \text{[using (ii)]}$$

$$= \int_V \frac{p}{\rho}\frac{d\rho}{dt}dV = \int_V \frac{p}{\rho}\left(-\rho\vec{\nabla}\cdot\vec{V}\right)dV \left(\because \frac{d\rho}{dt} + \rho\vec{\nabla}\cdot\vec{V} = 0, \text{ equation of continuity}\right)$$

i.e., $\frac{dI}{dt} = -\int_V p\left(\vec{\nabla}\cdot\vec{V}\right)dV$

Hence, finally we have

$$\frac{dT}{dt} = \int_V \vec{V}\cdot\frac{d\vec{V}}{dt}\rho dV \quad \text{... (iii)}$$

$$\frac{dW}{dt} = \int_V \frac{d\Omega}{dt}\rho dV, \quad \text{... (iv)}$$

and $\frac{dI}{dt} = -\int_V p\left(\vec{\nabla}\cdot\vec{V}\right)dV$ ... (v)

Again the Euler's equation of motion for conservative field is

$$\frac{d\vec{V}}{dt} = -\vec{\nabla}\Omega - \frac{1}{\rho}\vec{\nabla}p$$

So, $\vec{V}\cdot\frac{d\vec{V}}{dt} = \vec{V}\cdot\left[-\vec{\nabla}\Omega - \frac{1}{\rho}\vec{\nabla}p\right] = -\left(\vec{V}\cdot\vec{\nabla}\right)\Omega - \frac{1}{\rho}\vec{V}\cdot\vec{\nabla}p$

or, $\vec{V}\cdot\frac{d\vec{V}}{dt}\rho dV = -\left[\left(\vec{V}\cdot\vec{\nabla}\right)\Omega\right]\rho dV - \left(\vec{V}\cdot\vec{\nabla}p\right)dV$

Integrating over V,

$$\int_V \vec{V}\cdot\frac{d\vec{V}}{dt}\rho dV = -\int_V(\vec{V}\cdot\vec{\nabla})\Omega\rho dV - \int_V(\vec{V}\cdot\vec{\nabla}p)dV$$

or, $\frac{dT}{dt} = -\int_V \frac{d\Omega}{dt}\rho dV - \int_V(\vec{V}\cdot\vec{\nabla}p)dV$ [using (i) and (iii)]

or, $\frac{dT}{dt} = -\frac{dW}{dt} - \int_V(\vec{V}\cdot\vec{\nabla}p)dV$ [using (iv)]

or, $\frac{dT}{dt} + \frac{dW}{dt} = -\int_V(\vec{V}\cdot\vec{\nabla}p)dV$ ... (vi)

But we know that the vector identity

$$\vec{\nabla}\cdot(p\vec{V}) = p\vec{\nabla}\cdot\vec{V} + (\vec{V}\cdot\vec{\nabla})p$$

$$\therefore \int_V \vec{\nabla}\cdot(p\vec{V})dV = \int_V p(\vec{\nabla}\cdot\vec{V})dV + \int_V(\vec{V}\cdot\vec{\nabla})pdV$$

or, $\int_S -\vec{n}\cdot(p\vec{V})dS = -\frac{dI}{dt} + \int_V(\vec{V}\cdot\vec{\nabla})pdV$ (using Gauss's div. theorem and (v), $\vec{n}$ is inward normal)

or, $-\int_S \vec{n}\cdot(p\vec{V})dS + \frac{dI}{dt} = \int_V(\vec{V}\cdot\vec{\nabla})pdV$ ... (vii)

Using (vii) in (vi) we get

$$\frac{dT}{dt} + \frac{dW}{dt} = -\left[-\int_S \vec{n}\cdot(p\vec{V})dS + \frac{dI}{dt}\right]$$

or, $\frac{dT}{dt} + \frac{dW}{dt} + \frac{dI}{dt} = \int_S \vec{n}\cdot(p\vec{V})dS$ ... (viii)

which is known as equation of energy of a perfect fluid.

Corollary 1: Energy equation for incompressible fluid.

Proof. In case of incompressible fluid we have

$$\vec{\nabla}\cdot\vec{V} = 0$$

So, from (v), we get

$$\frac{dI}{dt} = 0$$

Hence from (viii) we have

$$\frac{dT}{dt}+\frac{dW}{dt}=\int_S \vec{n}\cdot\left(p\vec{V}\right)dS \text{ .... (ix)}$$

which is the equation of energy for incompressible fluid.

Corollary 2: Energy equation for incompressible fluid within the fixed boundary.

Proof. If the fluid is bounded on all sides by fixed boundary so that the fluid moves only tangentially over the surface, i.e., there is no normal flow across the boundary, then $\vec{n}\cdot\vec{V}=0$ at even point on S.

Now from (ix) i.e., equation of energy for incompressible fluid, we get

$$\frac{dT}{dt}+\frac{dW}{dt}=\int_S \vec{n}\cdot\left(p\vec{V}\right)dS=0$$

i.e. $$\frac{d\left(T+W\right)}{dt}=0$$

or, $T+W=$ contsnt

Hence, if incompressible fluid is contained within a fixed boundary, the sum of its kinetic and potential energies remain unchanged with the passage of time.

6.13 Kelvin's Minimum Energy Theorem:

The irrotational motion of an incompressible fluid occupying a simply connected region has less kinetic energy than any other motion of the fluid for which fluid has on the boundary same normal velocity as irrotational motion.

Proof. Let us consider an incompressible fluid occupying a simply-connected region V bounded by the closed surface S. Let ρ be density of the fluid, T the kinetic energy of the fluid moving irrotationally in which u, v, w are components of fluid velocity. Then

$$T=\int \frac{1}{2}\left|\vec{V}\right|^2 \rho dV=\frac{1}{2}\rho\int\left(u^2+v^2+w^2\right)dV \text{ ...... (i)}$$

Since the fluid motion is irrotational with velocity potential φ, therefore,

$$\vec{V}=-\vec{\nabla}\varphi$$

i.e. $$u=-\frac{\partial\varphi}{\partial x},\quad v=-\frac{\partial\varphi}{\partial y},\quad w=-\frac{\partial\varphi}{\partial z} \text{ ...... (ii)}$$

at everywhere.

Again, the equation of continuity for incompressible fluid is

$$\vec{\nabla}\cdot\vec{V}=0$$

i.e. $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad \text{...... (iii)}$$

at every interior point.

Let T' be the kinetic energy of any other possible stage of motion of the fluid in which velocity components are u', v', w'. Then we must have

$$T'=\frac{1}{2}\rho\int_V\left(u'^2+v'^2+w'^2\right)dV \quad \text{..... (iv)}$$

where velocity components satisfy the equation of continuity

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0 \quad \text{....... (v)}$$

at every interior point.

Since these two motions have the same normal velocity on the boundary surface S then we must have

$$\vec{n}\cdot\vec{V}=\vec{n}\cdot\vec{V}'$$

i.e. $$n_1u+n_2v+n_3w=n_1u'+n_2v'+n_3w' \quad \text{....... (vi)}$$

where $\vec{n}=(n_1,n_2,n_3)$.

Now $$T'-T=\frac{1}{2}\rho\int_V\left|\vec{V}'\right|^2dv-\frac{1}{2}\rho\int_V\left|\vec{V}\right|^2dv$$

$$=\frac{1}{2}\rho\int_V\left(\left|\vec{V}'\right|^2-\left|\vec{V}\right|^2\right)dv$$

$$=\frac{1}{2}\rho\int_V\left\{\left(u'^2-u^2\right)+\left(v'^2-v^2\right)+\left(w'^2-w^2\right)\right\}dV$$

$$=\frac{1}{2}\rho\int_V\left[\left(u'-u\right)^2+2u\left(u'-u\right)+\left(v'-v\right)^2+2v\left(v'-v\right)+\left(w'-w\right)^2+2w\left(w'-w\right)\right]dV$$

$$=\frac{1}{2}\rho\int_V\left[\left(u'-u\right)^2+\left(v'-v\right)^2+\left(w'-w\right)^2\right]dV+\rho\int_V\left[u\left(u'-u\right)+v\left(v'-v\right)+\left(w'-w\right)\right]dV$$

$$=I_1+I_2,\ \text{say} \quad \text{............. (vii)}$$

where, $$I_1=\frac{1}{2}\rho\int_V\left[\left(u'-u\right)^2+\left(v'-v\right)^2+\left(w'-w\right)^2\right]dV=\frac{1}{2}\rho\int_V\left|\vec{V}'-\vec{V}\right|^2dV$$

and $\quad I_2 = \rho \int_V [u(u'-u)+v(v'-v)+w(w'-w)]dV$

Now, I_2 can be simplified in the following way:

$$I_2 = \rho \int_V [u(u'-u)+v(v'-v)+w(w'-w)]dV$$

$$= \rho \int_V \vec{V}\cdot(\vec{V}'-\vec{V})dV$$

$$= \rho \int_V \left[-\frac{\partial \varphi}{\partial x}(u'-u) - \frac{\partial \varphi}{\partial y}(v'-v) - \frac{\partial \varphi}{\partial z}(w'-w) \right] dV \text{ [using (ii)]}$$

$$= -\rho \int_V \left[\frac{\partial}{\partial x}\{\varphi(u'-u)\} + \frac{\partial}{\partial y}\{\varphi(v'-v)\} + \frac{\partial}{\partial z}\{\varphi(w'-w)\} \right] dV$$

$$+\rho \int_V \varphi \left[\frac{\partial}{\partial x}(u'-u) + \frac{\partial}{\partial y}(v'-v) + \frac{\partial}{\partial z}(w'-w) \right] dv \quad \text{...... (viii)}$$

$$= -\rho \int_V \left[\frac{\partial}{\partial x}\{\varphi(u'-u)\} + \frac{\partial}{\partial y}\{\varphi(v'-v)\} + \frac{\partial}{\partial z}\{\varphi(w'-w)\} \right] dV \quad \text{[using (iii) and (v)]}$$

$$= -\rho \int_V \vec{\nabla}\cdot\{\varphi(\vec{V}'-\vec{V})\} dV$$

$$= -\rho \int_S \vec{n}\cdot\{\varphi(\vec{V}'-\vec{V})\} dS \text{ (by Gauss's div. theorem)}$$

$$= -\rho \int_S \varphi\{\vec{n}\cdot(\vec{V}'-\vec{V})\} dS$$

$$= -\rho \int_S \varphi\{n_1(u'-u)+n_2(v'-v)+n_3(w'-w)\} dS$$

$$= -\rho \int_S \varphi \cdot 0 dS \text{ [using (vi)]}$$

$$= 0$$

Hence, $\quad T'-T = I_1 = \frac{1}{2}\rho \int_V |\vec{V}'-\vec{V}|^2 dV$ = a positive quantity

$\therefore T'-T > 0$

or, $\quad T < T'$. Hence the theorem.

6.14 Uniqueness Theorem :

There cannot be two different forms of irrotational motion for a given confined mass of liquid whose boundaries have prescribed velocities.

Proof. If possible, let there be two such different irrotational motions. Let φ_1 and φ_2 be two different velocity potentials corresponding to two different irrotational motions. Therefore φ_1, φ_2 satisfy the Laplace equation, i.e.,

$$\nabla^2\varphi_1 = 0, \nabla^2\varphi_2 = 0 \text{ at every interior point} \ldots\ldots \text{(i)}$$

Since the boundaries have the same prescribed normal velocities, in each motion, so

$$\frac{\partial\varphi_1}{\partial n} = \frac{\partial\varphi_2}{\partial n} \text{ at every point on boundary} \ldots\ldots \text{(ii)}$$

Let $\varphi = \varphi_1 - \varphi_2$ .... (iii)

$$\therefore \nabla^2\varphi = \nabla^2\varphi_1 - \nabla^2\varphi_2 = 0 \text{ [using (i)]} \ldots\ldots \text{(iv)}$$

which shows that φ is a possible velocity potential of an irrotational motion in which kinetic energy is

$$T = \frac{1}{2}\rho\int_V \vec{V}\cdot\vec{V}dV = \frac{1}{2}\rho\int_V \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi dV \left(\because \vec{V} = -\vec{\nabla}\varphi \,\&\, \vec{\nabla}^2\varphi = 0\right)$$

$$= \frac{1}{2}\rho\int_V\left[\vec{\nabla}\cdot\left(\varphi\vec{\nabla}\varphi\right) - \varphi\nabla^2\varphi\right]dV\left(\because \vec{\nabla}\cdot\left(\varphi\vec{\nabla}\varphi\right) = \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + \varphi\vec{\nabla}^2\varphi\right)$$

$$= \frac{1}{2}\rho\int_V \vec{\nabla}\cdot\left(\varphi\vec{\nabla}\varphi\right)dV$$

$$= -\frac{1}{2}\rho\int_S \vec{n}\cdot\left(\varphi\vec{\nabla}\varphi\right)dS \text{ (by Gauss's div. theorem)}$$

$$= -\frac{1}{2}\rho\int_S \varphi\left(\vec{n}\cdot\vec{\nabla}\varphi\right)dS$$

$$= -\frac{1}{2}\rho\int_S \varphi\frac{\partial\varphi}{\partial n}dS$$

But $\dfrac{\partial\varphi}{\partial n} = \dfrac{\partial\varphi_1}{\partial n} - \dfrac{\partial\varphi_2}{\partial n} = 0$ at every point on boundary.

$$\therefore T = 0$$

Hence $\dfrac{1}{2}\rho\int_V\left|\vec{V}\right|^2 dV = 0$

or, $\int_V\left(u^2+v^2+w^2\right)dV = 0$ where $\vec{V} = (u, v, w)$

or, $\int_V\left[\left(\dfrac{\partial\varphi}{\partial x}\right)^2 + \left(\dfrac{\partial\varphi}{\partial y}\right)^2 + \left(\dfrac{\partial\varphi}{\partial z}\right)^2\right]dV = 0 \left(\because u = -\dfrac{\partial\varphi}{\partial x}, v = -\dfrac{\partial\varphi}{\partial y}, w = -\dfrac{\partial\varphi}{\partial z}\right)$

which $\Rightarrow \frac{\partial\varphi}{\partial x}=0, \frac{\partial\varphi}{\partial y}=0, \frac{\partial\varphi}{\partial z}=0$ everywhere

$\Rightarrow \varphi$ is independent of x, y, z

i.e., φ = constant everywhere.

$\therefore \varphi_1-\varphi_2$ = constant.

i.e., φ_1 and φ_2 can differ only by a constant. Therefore the velocity distribution given by φ_1 and φ_2 are identical and hence two motions are identical.

6.15 Motion in Two Dimensions :

Suppose a fluid moves in such a way that at any given instant the flow pattern in a certain plane is the same as that in all other parallel planes within the fluid. Then the flow is said to be two-dimensional.

If we take any one of the parallel planes to be the plane $z=0$, then at any point in the fluid having cartesian co-ordinates (x, y, z), all physical quantities (velocity, pressure, density, etc.) associated with the fluid are independent of z. Evidently in this case

$w=0$ and $u=u(x,y,t)$, $v=v(x,y,t)$ where $\vec{V}=(u,v,w)=(u,v,0)$.

6.16 Lagrange's Stream Function (Current function):

In case of two-dimensional motion, the differential equation of the stream lines are given by

$$\frac{dx}{u}=\frac{dy}{v}$$

i.e. $vdx-udy=0$ .... (i)

The equation of continuity for incompressible fluid, in two dimensions is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \text{.................. (ii)}$$

i.e. $\frac{\partial v}{\partial y}=\frac{\partial(-u)}{\partial x}$

Above result shows that (i) is an exact differential and let it will be $d\psi$, i.e.,

$$vdx-udy=d\psi=\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy$$

which $\Rightarrow v=\frac{\partial\psi}{\partial x}, -u=\frac{\partial\psi}{\partial y}$ .... (iii)

Now, (i) takes the form as

$$d\psi = 0$$

Int., ψ = constant .... (iv)

This function $\psi = \psi(x, y)$ is called the ***stream function or current function;***

Since the stream lines are given by (i), so it follows that stream function is constant along the stream line.

Note-1. Stream function exists for all types of two dimensionsl motion-rotational or irrotational.

Note-2. The necessary conditions for the existence of ψ are:

i) the flow must be continuous,

ii) the flow must be incompressible.

Note-3. The existence of a stream function is a consequence of stream lines and equation of continuity for incompressible fluid.

Note-4. φ and ψ are conjugate functions.

Proof. For irrotational fluid motion we have

$$\vec{V} = -\vec{\nabla}\varphi \text{ where } \varphi \text{ is velocity potential}$$

$$\Rightarrow u = -\frac{\partial \varphi}{\partial x}, v = -\frac{\partial \varphi}{\partial y}$$

If ψ is a stream function, then

$$u = -\frac{\partial \psi}{\partial y}, \; v = \frac{\partial \psi}{\partial x}$$

$$\therefore \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \; \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

So, $$\nabla^2\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{\partial \varphi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right) = 0$$

$\nabla^2\psi = 0$ .... (i)

Again, $u = -\frac{\partial \varphi}{\partial x}, v = -\frac{\partial \varphi}{\partial y}$ and equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

$$\therefore \frac{\partial}{\partial x}\left(-\frac{\partial \varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial \varphi}{\partial y}\right)=0$$

$$-\left[\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}\right]=0.$$

i.e. $\nabla^2\varphi=0$ .... (ii)

(i) and (ii) implies that φ and ψ both satisfies Laplace's equation i.e., φ and ψ are conjugate functions.

Note-5 Existence of φ and ψ :

i) The stream function ψ exists whether the motion is irrotational or not.

ii) The velocity potential φ exists only when the motion is irrotational.

iii) When motion is irrotational. φ exists.

iv) φ and ψ both satisfy Laplace's equation and

$$\frac{\partial \varphi}{\partial x}=\frac{\partial \psi}{\partial y},\ \frac{\partial \varphi}{\partial y}=-\frac{\partial \psi}{\partial x}.$$

Note-6 The family of curves $\varphi(x,y)=$ constant and $\psi(x,y)=$ constant, cut orthogonally at their points of intersection.

Proof. $\varphi(x,y)=$ constant $d\varphi=0$

i.e., $\dfrac{\partial \varphi}{\partial x}dx+\dfrac{\partial \varphi}{\partial y}dy=0$

$$\frac{dy}{dx}=-\frac{\varphi_x}{\varphi_y}=m_1\text{, say}$$

which is the gradient of tangent to the curve $\phi=$ constant.

Again, $\psi(x,y)=$ constant $\Rightarrow d\psi=0$

i.e. $\dfrac{\partial \psi}{\partial x}dx+\dfrac{\partial \psi}{\partial y}dy=0$

$$\therefore \frac{dy}{dx}=-\frac{\psi_x}{\psi_y}=m_2\text{, say}$$

which is the gradient of tangent to the curve $\psi=$ constant.

Now, $m_1 \times m_2 = \left(-\frac{\varphi_x}{\varphi_y}\right)\times\left(-\frac{\psi_x}{\psi_y}\right) = \frac{u}{v}\times\frac{v}{(-u)} = -1$

Hence the curves of constant potential and constant stream functions cut orthogonally at their points of intersection.

6.17 Complex Potential:

At this stage we introduce the notion of a function of a complex variable. Suppose φ and ψ represent velocity potential and stream function of a two dimensional irrotational motion of a perfect fluid. Let

$$w = f(z) = \varphi(x,y) + i\psi(x,y) \text{ where } z = x+iy, i = \sqrt{-1}.$$

Then w is defined as complex potential of the fluid.

Property 1. w is an analytic function.

Proof. Since $w = f(z) = \varphi(x,y) + i\psi(x,y)$ where $z = x+iy$

So, we must have

$$u = -\frac{\partial\varphi}{\partial x},\ v = -\frac{\partial\varphi}{\partial y}$$

and $u = -\frac{\partial\psi}{\partial y},\ v = \frac{\partial\psi}{\partial x}$.

where u and v are velocity components of the fluid.

$$\therefore \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \text{ and } \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

which are Cauchy-Riemann equations. Thus Cauchy-Riemann equation are satisfied so that w is analytic function of z.

Conversely if w is analytic function, then its real and imaginary i.e., φ and ψ gives the velocity potential and stream function for a possible two dimensional irrotational fluid motion.

Property 2. $\left|\frac{dw}{dz}\right|$ is the magnitude of the velocity of the fluid at any point.

Proof. Since $w = f(z) = \varphi(x,y) + i\psi(x,y), z = x+iy$

$$\therefore \frac{dw}{dz} = f'(z) = \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{Lt}\left[\frac{\delta\varphi + i\delta\psi}{\delta x + i\delta y}\right]$$

where $\delta\varphi = \varphi(x+\delta x, y+\delta y) - \varphi(x,y)$

$\delta\psi = \psi(x+\delta x, y+\delta y) - \psi(x,y)$

At first we keep y as a constant, then

$$\frac{dw}{dz} = \underset{\delta x \to 0}{Lt}\left[\frac{\delta\varphi + i\delta\psi}{\delta x}\right] = \frac{\partial\varphi}{\partial x} + i\frac{\partial\psi}{\partial x} = -u + iv$$

where $u = -\frac{\partial\varphi}{\partial x} = -\frac{\partial\psi}{\partial y}, v = -\frac{\partial\varphi}{\partial y} = \frac{\partial\psi}{\partial x}$.

Similarly, if we keep x as a constant then

$$\frac{dw}{dz} = \underset{\delta y \to 0}{Lt}\left[\frac{\delta\varphi + i\delta\psi}{i\delta y}\right] = -i\frac{\partial\varphi}{\partial y} + \frac{\partial\psi}{\partial y} = -u + iv$$

Hence, $\left|\frac{dw}{dz}\right| = \sqrt{u^2 + v^2}$ = magnitude of the velocity.

Therefore $\left|\frac{dw}{dz}\right|$ represents the magnitude of the velocity.

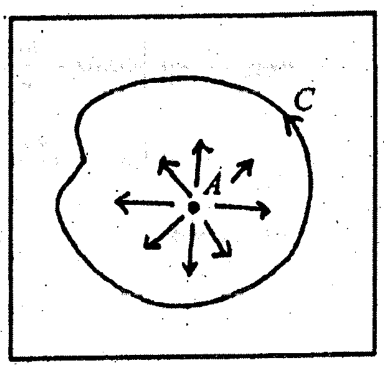
Stagnation Points: The points, where the fluid velocity is zero, are called stagnation points.

Thus for stagnation points, we must have

$$\frac{dw}{dz} = 0.$$

6.18 Two Dimensional Sources, Sinks:

i) **Sources:** A source is a point from which liquid is emitted radially and symmetrically in all directions in xy-plane.



ii) **Sink:** A point to which fluid is flowing in symmetrically and radially in all directions is called sink. This sink is a negative of source.

Source is a point at which liquid is continuously created and sink is a point at which liquid is continuously annihilated. Really speaking, source and sink are purely abstract conceptions which do not occur in nature.

iii) **Strength:** Strength of a source is defined as total volume of flow per unit time from it.

Thus if $2\pi m$ is the total volume of flow across any small circle surrounding the source, then m is called strength of the source. Sink is a source of strength $(-m)$.

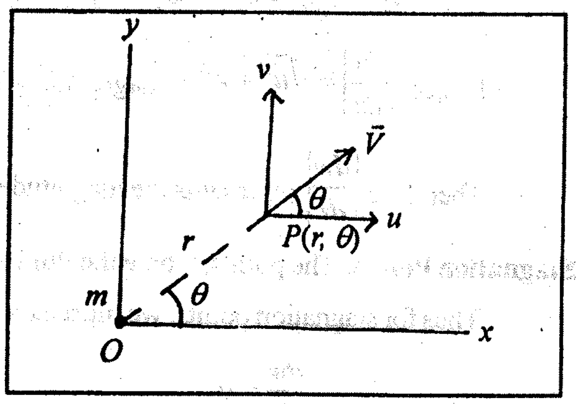
If we consider, for C the circle centre at A of radius r, the speed of flow $|\vec{V}|$ is everywhere of the same on C and the mass is now $2\pi r\rho|\vec{V}|$ per unit length. Thus

$$(2\pi m)\rho = 2\pi r\rho|\vec{V}| \text{ for unit length}$$

$$\therefore |\vec{V}| = \frac{m}{r} \text{ or, } -\frac{\partial\varphi}{\partial r} = \frac{m}{r}.$$

6.19 Complex Potential Due to a Source:

Let us consider a source of strength m at the origin. Let w be the complex potential at $P(r,\theta)$ due to this source. The velocity at P due to the source is purely radial, i.e., along $\overline{OP}$ and let this velocity be $|\vec{V}|$. Now the flux across a circle of radius r surrounding the source at O is $2\pi r|\vec{V}|$. So



$$2\pi r|\vec{V}| = 2\pi m$$

$$\Rightarrow |\vec{V}| = \frac{m}{r}$$

Let u, v be the velocity components along Ox and Oy,

then $u = |\vec{V}|\cos\theta = \frac{m}{r}\cos\theta$

$v = |\vec{V}|\sin\theta = \frac{m}{r}\sin\theta$

Again, we know that,

$$\frac{dw}{dz} = -u + iv = \frac{m}{r}[-\cos\theta + i\sin\theta] = -\frac{m}{r}e^{i\theta}$$

$$= -\frac{m}{re^{i\theta}} = -\frac{m}{z}$$

$$\therefore dw = -m\frac{dz}{z}$$

Integrating, $w = -m\log_e z$ (Neglecting constant of integration).

If the source of strength m is at a point $z = z_1$ in place of $z=0$, then by shifting the origin, we have

$$w = -m\log_e(z - z_1).$$

Corollary: Complex potential at a point due to the sources of strengths $m_1, m_2, m_3, \ldots$ situated at $z_1, z_2, z_3, \ldots$

Proof. We have the complex potential w due to a source of strength m situated at $z = z_1$ is

$$w = -m\log_e(z - z_1)$$

Hence the required complex potential due to the sources of strengths $m_1, m_2, m_3, \ldots$ situated at $z_1, z_2, z_3, \ldots$ is

$$w = -m_1\log_e(z - z_1) - m_2\log_e(z - z_2) - m_3\log_e(z - z_3) - \ldots\ldots$$

$$= -\sum_{j=1}^{\infty} m_j \log_e(z - z_j)$$

which $\Rightarrow \varphi = -\sum_{j=1}^{\infty} m_n \log_e r_n$

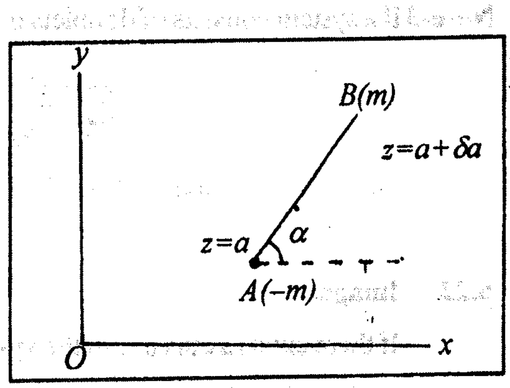
$$\psi = -\sum_{n=1}^{\infty} m_n\theta_n, \text{ where } z - z_n = r_n e^{i\theta_n}.$$

6.20 Doublet:

A doublet is defined as a combination of the equal and opposite sources, i.e., a source and a sink of strengths $(+m)$ and $(-m)$ respectively, situated at a distance δs apart such that the product $m\delta s$ is finite.

Strength of doublet: If $m\delta s = \mu$ = finite, where $m \to \infty$, $\delta s \to 0$ then μ is called strength of the doublet and line δs is called the axis of the doublet and its direction is taken from sink to source.

6.21 Complex Potential for a Doublet:



Let μ be the strength of a doublet AB formed by a sink $-m$ at $A(z = a)$ and a source $+m$ at $B(z = a + \delta a)$. Then

$$\left.\begin{aligned}\mu &= m.AB\\ \text{and } \delta a &= ABe^{i\alpha}\end{aligned}\right\} \ldots\ldots(i) \quad \left(\because z = re^{i\theta}\right)$$

where α is the inclination of the axis of the doublet with Ox.

If w be the complex potential due to this doublet at any point $P(z)$ then

$$w = +m\log(z-a) - m\log\left(z-\overline{a+\delta a}\right)$$

$$= -m\log_e\left(\frac{z-a-\delta a}{z-a}\right)$$

$$= -m\log_e\left(1-\frac{\delta a}{z-a}\right)$$

$$= -m\left[-\frac{\delta a}{z-a} - \left(\frac{\delta a}{z-a}\right)\cdot\frac{1}{2} - \ldots\ldots\right]$$

$$= +m\left(\frac{\delta a}{z-a}\right) \text{ (upto first approximation)}$$

$$= \frac{m\, AB.e^{i\alpha}}{z-a}$$

$$= \frac{\mu e^{i\alpha}}{z-a} \text{ [using (i)]}$$

$$\therefore w = \mu\frac{e^{i\alpha}}{z-a}$$

Note-1 If the axis of the doublet is along x-axis, then $\alpha = 0^0$

so that $$w = \frac{\mu}{z-a}.$$

Note-2 If the axis of the doublet is along x-axis and the doublet is at the origin, then $\alpha = 0^0, a = 0$, so that

$$w = \frac{\mu}{z}$$

Note-3 If a system consists of doublets of strengths $\mu_1, \mu_2, \ldots$ placed at $z = a_1, a_2, \ldots$, then

$$w = \sum_{n=1}^{\infty}\frac{\mu_n e^{i\alpha_n}}{z-a_n}$$

where α_n is the inclination of the axis of the doublet of strength μ_n with Ox.

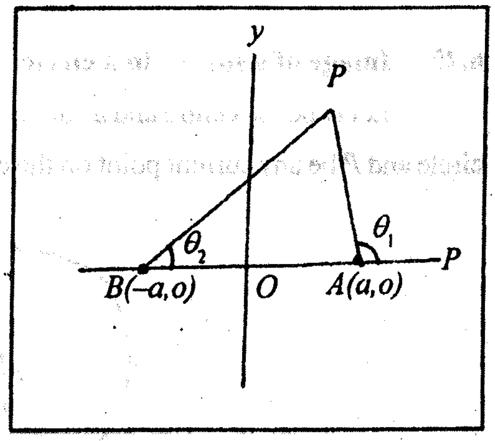
6.22 Image:

If there exists a curve C in the xy-plane in a fluid such that there is no flow across it, then the system of sources, sinks and doublets on one side of C is said to be the image of the sources, sinks and doublets on other side of C.

In two-dimensional irrotational motion when confined to rigid boundaries is regarded to have been caused by the presence of sources and sinks. If we take the set of sources and sinks to be on either side of the rigid boundaries, the velocity normal to these boundaries will be zero. As such these boundaries can be taken as stream lines. This is due to the property of stream lines that the velocity perpendicular to stream lines is zero. This set of sources and sinks on either side is called the image. Thus the motion is no longer constrained by boundaries so that it is possible to predict the nature of the velocity and pressure at each point of the fluid.

6.23 Image of a source w.r.t. a strength line (plane):

To determine the image of a source $+m$ at $A(a,0)$ w.r.t. the line Oy we place a source $+m$ at $B(-a,0)$. Then complex potential w at any point $P(z)$ due to this system is



$$w = -m\log_e(z-a) - m\log_e(z+a)$$

$$= -m\log_e(z-a)(z+a)$$

$$= -m\log_e\left\{r_1 r_2 e^{i(\theta_1+\theta_2)}\right\}$$

where $AP = r_1, BP = r_2, z-a = r_1e^{i\theta_1}, z+a = r_2e^{i\theta_2}$.

$$\therefore \varphi + i\psi = -m\left\{\log_e r_1r_2 + i(\theta_1+\theta_2)\right\}$$

which $\Rightarrow \varphi = -m\log_e r_1r_2$

and $\psi = -m(\theta_1+\theta_2)$

If P lies on Oy, i.e., y-axis then $AP = BP$ so that

$$\angle PAB = \angle PBA$$

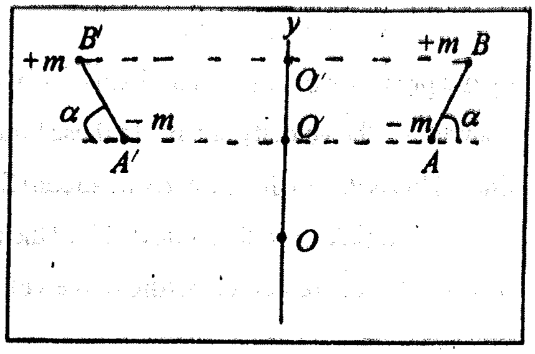
i.e. $\pi - \theta_1 = \theta_2$

$$\therefore \theta_1 + \theta_2 = \pi$$

Hence $\psi = -m\pi$ = constant.

Which implies that y-axis is a stream line. Hence the image of a source $+m$ at $A(a,0)$ is a source $+m$ at $B(-a,0)$, i.e., image of a source w.r.t. a line is a source of the same strength situated on the opposite side of the line at an equal distance.

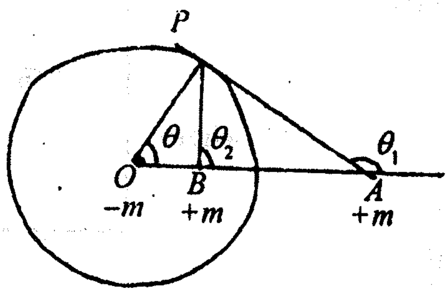
6.24 Image of a doublet w.r.t. a plane:



To find the image of the doublet we consider a pair of sources $-m$ at A, $+m$ at B, close together and on one side of the rigid plane Oy. The image system is $-m$ at A', $+m$ at B', where A', B' are the respective optic images of the points A, B in the plane $OO'y$ i.e., $O'A = O'A'$ and $O''B = O''B'$. Hence the image of a doublet in an infinite rigid plane is an equal doublet symmetrically disposed with respect to the plane.

6.25 Image of a source in a circle:

Let O be the centre and a source $+m$ at A (outside of the circle). Let B be the inverse point of A w.r.t. the circle and P be any current point on the circle at which ψ is to be determined.



Now we place a source $+m$ at B and sink $-m$ at O. Then complex potential is

$$w = -m\log r_1e^{i\theta_1} - m\log r_2e^{i\theta_2} + m\log re^{i\theta} \left(\text{where } AP = r_1, BP = r_2, OP = r\right)$$

$$= -m\left[\log\left(\frac{r_1r_2}{r}\right) + i\left(\theta_1 + \theta_2 - \theta\right)\right]$$

$$\Rightarrow \psi = -m\left(\theta_1 + \theta_2 - \theta\right) \text{ .... (i)}$$

Since, B is the inverse point of A, so

$$OB \cdot OA = (\text{radius})^2 = OP^2 = r^2$$

$$\therefore \frac{OB}{OP} = \frac{OP}{OA}$$

$$\Rightarrow m \cdot BB' = m \cdot A'A \cdot \frac{OB}{OA'} = (m \cdot A'A)\frac{OB \cdot OA}{OA' \cdot OA}$$

Taking limits as $A' \to A$, so that $B' \to B$. Then we get

$$\mu' = \mu \cdot \frac{a^2}{f^2}$$

Hence $\mu' = \frac{\mu a^2}{f^2}$.

Again from similarity of triangles, we have

$$\angle OBB' = \angle OA'A = \angle OAT = \alpha.$$

Thus the image of a doublet of strength μ at A relative to a circle is a doublet of strength $\mu' = \frac{\mu a^2}{f^2}$ at B where $OA = f$, the inverse point of A, the axis of the doublet make supplementary angles i.e., $\pi - \alpha$ with the radius OA.

6.27 Milne-Thomson's Circle Theorem:

Let $f(z)$ be the complex potential of a two-dimensional irrotational motion of an incompressible inviscid liquid having no rigid boundaries and such that there are no singularities of flow within the circle $|z| = a$. Then, on introducing the solid circular cylinder $|z| = a$ into the flow, the new complex potential of the resulting motion is given by

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) \quad \text{for} \quad |z| \geq a.$$

Proof. All the singularities of $f(z)$ (at which sources, sinks, doublets or vortices may be present) occur in the region $|z| = a$, and so the singularities of $f\left(a^2/z\right)$ lie in $|z| = a$. Hence the singularities of $\bar{f}\left(a^2/z\right)$ also lie in $|z| < a$. Thus $f(z)$ and $f(z) + \bar{f}\left(a^2/z\right)$ both have the same singularities in the region $|z| > a$ and so both functions, considered as complex velocity potentials, may be described to the same hydrodynamical distribution in the region $|z| > a$.

and $\angle BOP = \angle POA$

Threfore, ΔOPB and ΔOPA are similar. So,

$\angle OPB = \angle OAP$

i.e., $\theta_2 - \theta = \pi - \theta_1$

$\therefore \theta_2 + \theta_1 - \theta = \pi$ ....(ii)

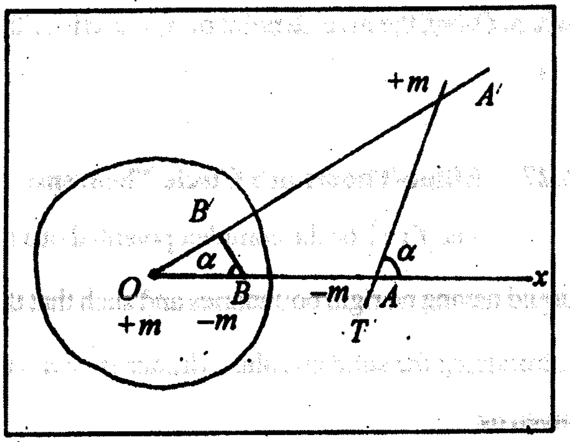
Hence, $\psi = -m\pi$ [using (i) and (ii)]

$\Rightarrow \psi =$ constant.

which shows that circle is a stream line so that there exists no flux across the boundary, that means, image of source $+m$ at A is a soruce $+m$ at B, the inverse point of A, and sink $-m$ at the centre.

6.26 Image of a doublet relative to a circle:

Let α be the inclination of the axis of the doublet AA' and O the centre of the circle. Let $OA = f$ and μ the strength of the doublet. Let us assume that this doublet as a combination of sink $-m$ and source $+m$ so that $\mu = m.AA'$ where $m \to \alpha$ and $AA' \to 0$. The image of sink $-m$ at A is a sink $-m$ at B; the inverse point of A and source $+m$ at O. The image of source $+m$ at A' is a source $+m$ at B'; the inverse point of A' and sink $-m$ at O.



Combining these, we find that source $+m$ and sink $-m$ both at O cancel each other and there remains a doublet of strength $\mu = m.BB'$ at B, the inverse of A.

In limit sense, $A' \to A \Rightarrow B' \to B$.

Now, $OB \cdot OA = a^2 = OB' \cdot OA'$, a is the radius of the circle

$$\therefore \frac{OB}{OB'} = \frac{OA'}{OA}$$

Also $\angle BOB' = \angle A'OA$

$\Rightarrow \Delta OBB'$ and $\Delta OAA'$ are similar

Hence, $\frac{BB'}{A'A} = \frac{OB}{OA'}$

On the circle $|z| = a$, we take $z = ae^{i\theta}$.

Then $\frac{a^2}{z} = ae^{-i\theta}$ and so

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$$

$$= f\left(ae^{i\theta}\right) + \bar{f}\left(ae^{-i\theta}\right)$$

$$= f\left(ae^{i\theta}\right) + \overline{f\left(ae^{i\theta}\right)}. \left(\because \bar{f}(\bar{z}) = \overline{f(z)}, \text{complex conjugate of } f(z)\right)$$

$$= f(z) + \overline{f(z)}.$$

Thus on $|z| = a$, w is the sum of a complex quantity and its complex conjugate and is therefore a real number.

i.e. w = real number

$\Rightarrow \psi = 0$ on $|z| = a$.

This shows that the circular boundary is a stream line across which no fluid flows. Hence $|z| = a$ is a possible boundary for the new flow and $w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$ is the appropriate complex velocity potential for the new flow.

6.28 Applications of the Circle Theorem:

i) *Uniform flow past a stationary cylinder:*

The uniform stream having velocity $-U\vec{i}$ gives rise to a complex potential Uz. So

$$f(z) = Uz$$

Then $\bar{f}(z) = Uz$

and so $\bar{f}\left(\frac{a^2}{z}\right) = \frac{Ua^2}{z}$

Thus on introducing the cylinder of circular section $|z| = a$ into the stream, the complex potential for the region $|z| \geq a$ becomes

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) = U\left(z + \frac{a^2}{z}\right)$$

Taking $z = re^{i\theta}$ and equating real and imaginary parts, we get

$$\varphi = U\cos\theta\left(r + \frac{a^2}{r}\right)$$

$$\psi = U\sin\theta\left(r - \frac{a^2}{r}\right)$$

ii) *Image of a source w.r.t. a circle*:

Let us consider a source of strength $+m$ at $z=f$. Then the complex potential due to this source is

$$f(z) = -m\log_e(z-f).$$

Let us introduce a circular cylinder $|z| = a(a < f)$, then the complex potential, by circular theorem, is

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) = -m\log_e(z-f) - m\log_e\left(\frac{a^2}{z} - f\right)$$

$$= -m\log_e(z-f) - m\log_e\left(\frac{a^2 - fz}{z}\right)$$

$$= -m\log_e(z-f) - m\log_e\left\{\left(-\frac{f}{z}\right)\left(\frac{a^2}{f} - z\right)\right\}$$

$$= -m\log_e(z-f) - m\log_e\left(z - \frac{a^2}{f}\right) - m\log_e(-f) + m\log_e z$$

$$= -m\log_e(z-f) - m\log_e\left(z - \frac{a^2}{f}\right) + m\log_e z$$

[ignoring constant term $-m\log_e(-f)$]

This is the complex potential due to

i) source $+m$ at $z=f$,

ii) sink $-m$ at $z=0$,

iii) source $+m$ at $z = a^2/f$.

For this complex potential, circle is a stream line and hence the image system for a source $+m$ outside the circle consists of a source $+m$ at the inverse point and sink $-m$ at the origin, the centre of the circle, since f and a^2/f both are inverse points w.r.t. the circle $|z| = a$.

iii) *Image of a doublet*:

The complex potential $f(z)$ due to the doublet of strength μ at $z=f$ with its axis inclined at an angle α, is given by

$$f(z) = \frac{\mu e^{i\alpha}}{z-f}.$$

When a circular cylinder $|z| = a$, where $a < f$, is inserted in the flow of motion, then the complex potential is given by

$$w = f(z) + \bar{f}\left(a^2/z\right)$$

$$= \frac{\mu e^{i\alpha}}{z-f} + \left(\frac{\overline{\mu e^{i\alpha}}}{\frac{a^2}{z} - f}\right)$$

$$= \frac{\mu e^{i\alpha}}{z-f} + \frac{\mu e^{i\alpha}}{\frac{a^2}{z} - f} = \frac{\mu e^{i\alpha}}{z-f} - \frac{\mu z e^{i(\alpha-\pi)}}{a^2 - fz}$$

$$= \frac{\mu e^{i\alpha}}{z-f} + \frac{\mu z e^{i(\pi-\alpha)}}{f\left[z - \frac{a^2}{f}\right]}$$

$$= \frac{\mu e^{i\alpha}}{z-f} + \frac{\mu e^{i(\pi-\alpha)} \cdot \left(z - \frac{a^2}{f} + \frac{a^2}{f}\right)}{f\left[z - \frac{a^2}{f}\right]}$$

$$= \frac{\mu e^{i\alpha}}{z-f} + \frac{\mu}{f} \cdot e^{i(\pi-\alpha)} + \frac{\mu a^2}{f^2} \cdot \frac{e^{i(\pi-\alpha)}}{\left(z - \frac{a^2}{f}\right)}$$

$$= \frac{\mu e^{i\alpha}}{z-f} + \frac{\mu a^2}{f^2} \cdot \frac{e^{i(\pi-\alpha)}}{z - \frac{a^2}{f}} \qquad \text{(ignoring constant term)}$$

This is the complex potential due to

i) doublet of strength μ at $z=f$ with its axis inclined at an angle α,

ii) doublet of strength $\frac{\mu a^2}{f^2}$ at $z = \frac{a^2}{y}$, the inverse point of $z=f$, its axis is inclined at an angle $\pi - \alpha$.

For this complex potential circle is a stream line and hence the image system for a doublet of strength μ at $z=f$ (outside of the circle) is a doublet of strength $\mu' = \frac{\mu a^2}{f^2}$ and its axis inclined at an angle $\pi - \alpha$.

6.29 Unit Summary :

In this module we have discussed the motion of perfect fluid in simple and straight-forward method. This portion have covered the remain portion of the module 53, and for complete knowledge about perfect fluid, this portin is essential for each and every student of Mathematics.

6.30 Worked Out Examples

1. *A sphere of radius a is surrounding by infinite liquids of density ρ, the pressure at infinite being Π. The sphere is suddenly annihilated. Show that pressure at a distance r from the centre immediately falls to $\Pi\left(1-\frac{a}{r}\right)$.*

Show further that if the liquid is brought to rest by impinging on a concentric sphere of radius $\frac{a}{2}$, the impulsive pressure sustained by the surface of this sphere is $\left[\frac{7}{6}\Pi\rho a^2\right]^{1/2}$.

Ans. The equation of continuity, for spherical symmetry, is

$$x^2 v = F(t)$$

$$\therefore \frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$$

The equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\therefore \frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right), \rho = \text{constant}$$

Int., w.r.t. 'x', $-\frac{F'(t)}{x} + \frac{v^2}{2} = -\frac{p}{\rho} + C_0$ ....(1)

Boundary conditions are

i) when $x \to \infty, p = \Pi, v = 0$;

ii) when $x = a, v = \dot{x} = 0, p = 0, t = 0$;

iii) when $t = 0, x = r, v = 0, p = p_0$, where $r > a$.

Using (i) and (ii) in (1), then we get

$$0 = -\frac{\Pi}{\rho} + C_0$$

and $-\frac{F'(0)}{a} + 0 = 0 + C_0$

which $\Rightarrow \frac{-F'(0)}{a} = C_0 = \frac{\Pi}{\rho}$ ....(2)

Again using (iii) in (1), we get

$$\frac{-F'(0)}{r} = -\frac{p_0}{\rho} + C_0$$

or, $\frac{a\Pi}{\rho}\cdot\frac{1}{r} = -\frac{p_0}{\rho} + \frac{\Pi}{\rho}$ [using (2)]

$$\therefore p_0 = \Pi\left(1 - \frac{a}{r}\right).$$

2nd Part: Let ϖ be the required impulsive pressure.

Let u be the velocity of the inner surface of radius r. Then when $x = r, v = \dot{r} = u, p = 0$ when $r < a$.

Since pressure vanishes on the inner surface, so from (1),

we get $$\frac{-F'(t)}{r}+\frac{1}{2}u^2=C_0=\frac{\Pi}{\rho}$$

or, $$\frac{-F'(t)}{r}+\frac{1}{2}\frac{F^2}{r^4}=\frac{\Pi}{\rho}\left[\because r^2u=F(t)\right]$$

or, $$\frac{-2FF'(t)dt}{r}+F^2\frac{dr}{r^2}=\frac{\Pi}{\rho}\cdot 2r^2dr\left[\because r^2\frac{dr}{dt}=F(t)\right]$$

or, $$d\left(\frac{-F^2}{r}\right)=\frac{\Pi}{\rho}2r^2dr$$

Int. $$\frac{-F^2}{r}=\frac{2}{3}\cdot\frac{\Pi}{\rho}\cdot r^3+C_1,\text{ i.e., }-r^3u^2=\frac{2}{3}\frac{\Pi}{\rho}r^3+C_1$$

Using condition (ii), in above result, then we get

$$0=\frac{2}{3}\frac{\Pi}{\rho}\cdot a^3+C_1$$

$$\therefore C_1=-\frac{2}{3}\frac{\Pi}{\rho}a^3$$

Hence, $$-r^3u^2=\frac{2}{3}\frac{\Pi}{\rho}\left(r^3-a^3\right)$$

or, $$u^2=\frac{2}{3}\frac{\Pi}{\rho}\left(\frac{a^3}{r^3}-1\right)$$

$$\therefore\left[u^2\right]_{r=\frac{a}{2}}=\frac{2}{3}\frac{\Pi}{\rho}\cdot 7$$

i.e., $$u_{r=\frac{a}{2}}=\left(\frac{14\Pi}{3\rho}\right)^{1/2}$$

Now the equation of impulsive action is

$$d\tilde{w}=\rho v dx$$

$$\Rightarrow d\tilde{w}=\rho u dr$$

$$\Rightarrow \int_0^{\tilde{w}} d\tilde{w} = \rho u_{r=a/2} \int_0^{a/2} dr$$

$$\therefore \tilde{w} = \rho \cdot \left(\frac{14\Pi}{3\rho}\right)^{1/2} \cdot \frac{a}{2} = \left[\frac{7}{6}\rho\Pi a^2\right]^{1/2}$$

Ex.2 *An infinite mass of fluid acted on by a force $\mu r^{\frac{-3}{2}}$ per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere $r = c$ in it, show that the cavity will be filled up after an interval of time $\left(\frac{2}{5\mu}\right)^{1/2} c^{5/4}$.*

Ans. If v and p be the velocity and pressure at a distance x from the origin, then the equation of continuity is

$$x^2 v = F(t)$$

i.e. $$v = \frac{F(t)}{x^2}$$

and the equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\mu x^{-3/2} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\therefore \frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) = -\mu x^{-3/2} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Int., $$-\frac{F'(t)}{x} + \frac{v^2}{2} = \frac{2\mu}{\sqrt{x}} - \frac{p}{\rho} + C_0 \quad \text{..............} (1)$$

Boundary conditions are :

i) when $x \to \infty, v = 0, p = 0$;

ii) when $x = r$ (radius of cavity), $p = 0, \dot{v} = \dot{r}$;

iii) when $x = c,\ v = 0$ so that $F(t) = 0$.

Using the conditions (i) and (ii) in (1) then we get

$$0 + 0 = 0 - 0 + C_0 \quad i.e.,\ C_0 = 0$$

and $$\frac{-F'(t)}{r} + \frac{1}{2}\dot{r}^2 = \frac{2\mu}{\sqrt{r}} - 0 + C_0$$

i.e. $\dfrac{-F'(t)}{r}+\dfrac{1}{2}\dot{r}^2=\dfrac{2\mu}{\sqrt{r}}$ .... (2)

Since $v=\dfrac{F(t)}{x^2}$ therefore $r^2\dot{r}=F(t)$ i.e., $r^2dr=F(t)dt$.

Hence from (2), we get by multiplying $2F(t)dt$ or $2r^2dr$ on both sides,

$$\frac{-2F(t)F'(t)dt}{r}+\frac{F^2(t)}{r^4}\cdot r^2dr=\frac{4\mu}{\sqrt{r}}\cdot r^2dr$$

or, $$d\left[\frac{-F^2(t)}{r}\right]=r\mu r^{3/2}dr$$

Int. $$\frac{-F^2(t)}{r}=4\mu\cdot\frac{2}{5}r^{5/2}+C_1 \text{ .... (3)}$$

Using boundary condition (iii) in (3), we get

$$0=\frac{8\mu}{5}\cdot c^{5/2}+C_1$$

$$\therefore C_1=-\frac{8\mu}{5}c^{5/2}$$

Hence from (3) we get

$$\frac{-(r^2\dot{r})^2}{r}=\frac{8\mu}{5}\left(r^{5/2}-c^{5/2}\right)$$

$$\Rightarrow\frac{dr}{dt}=-\left[\frac{8\mu}{5r^3}\left(c^{5/2}-r^{5/2}\right)\right]^{1/2}$$

[negative sign is taken as velocity increases when r decreases]

Let T be the required time of falling the cavity. Then integrating above with proper limits.

$$-\int_c^0\frac{r^{3/2}}{\sqrt{\left(c^{5/2}-r^{5/2}\right)}}dr=\int_0^T\frac{8\mu}{5}^{1/2}dt$$

$$\therefore T=\left(\frac{5}{8\mu}\right)^{1/2}\int_0^c\frac{r^{3/2}}{\sqrt{\left(c^{5/2}-r^{5/2}\right)}}$$

$$=\left(\frac{5}{8\mu}\right)^{1/2}\int_0^{\pi/2}\frac{4}{5}c^{5/2}\cdot\frac{\sin\theta\cos\theta d\theta}{c^{5/4}\cos\theta}\quad\left[\text{Put } r^{5/2}=c^{5/2}\sin\theta\right]$$

$$=\left(\frac{2}{5\mu}\right)^{1/2}\cdot c^{5/4}.$$

Ex-3. *Air, obeying Boyle's law, is in motion in a uniform tube of small section, prove that if* ρ *be the density and* v *the velocity at a distance* x *from the fixed point at time* t*, then* $\frac{\partial^2\rho}{\partial t^2}=\frac{\partial^2}{\partial x^2}\left[\left(v^2+k\right)\rho\right]$*, where* $p=k\rho$.

Ans. The equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0 \quad \text{.........(i)}$$

and equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{..................(ii)}$$

From Boyle's law, we have

$$p=k\rho$$

$$\therefore \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{k}{\rho}\frac{\partial\rho}{\partial x} \quad \text{...................(iii)}$$

Now, $\frac{\partial^2\rho}{\partial t^2}=\frac{\partial}{\partial t}\left(-\frac{\partial}{\partial x}(\rho v)\right)$ [using (i)]

$$=-\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}(\rho v)\right]$$

$$=-\frac{\partial}{\partial x}\left[v\frac{\partial\rho}{\partial t}+\rho\frac{\partial v}{\partial t}\right]$$

$$=-\frac{\partial}{\partial x}\left[v\left(-\frac{\partial}{\partial x}(\rho v)\right)+\rho\left(-\frac{k}{\rho}\frac{\partial\rho}{\partial x}-v\frac{\partial v}{\partial x}\right)\right]$$

$$=\frac{\partial}{\partial x}\left[v\frac{\partial}{\partial x}(\rho v)+k\frac{\partial\rho}{\partial x}+\rho v\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial (\rho v.v)}{\partial x} + \frac{\partial (k\rho)}{\partial x}\right]$$

$$= \frac{\partial^2}{\partial x^2}\left(\rho v^2 + k\rho\right)$$

$$\therefore \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[\rho\left(v^2 + k\right)\right]$$

Ex. 4 *An elastic fluid, the weight of which is neglected obeying Boyle's is in motion in a uniform straight tube; show that on the hypothesis of parallel sections the velocity at any time t at a distance r from a fixed point in the tube is defined by the equation* $\frac{\partial^2 v}{\partial t^2} + \frac{\partial}{\partial r}\left(v^2\frac{\partial v}{\partial r} + 2v\frac{\partial v}{\partial t}\right) = k\frac{\partial^2 v}{\partial r^2}$.

Ans. From Boyle's law, we have

$$p = k\rho$$

The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial r} = 0 \quad \text{................ (i)}$$

and the equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

i.e. $$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{k}{\rho}\frac{\partial \rho}{\partial r} \quad \text{............. (ii)}$$

$$\therefore \frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial r} - \frac{k}{\rho}\frac{\partial \rho}{\partial r}$$

So, $$\therefore \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial t}\right) = \frac{\partial}{\partial t}\left[-v\frac{\partial v}{\partial r} - \frac{k}{\rho}\frac{\partial \rho}{\partial r}\right]$$

$$= -\frac{\partial}{\partial t}\left[\frac{\partial}{\partial r}\left(\frac{v^2}{2}\right) + k\frac{\partial \log_e \rho}{\partial r}\right]$$

$$= -\frac{\partial}{\partial t}\left[\frac{\partial}{\partial r}\left(\frac{v^2}{2} + k^{\log_e \rho}\right)\right]$$

$$= -\frac{\partial}{\partial r}\left[\frac{\partial}{\partial t}\left(\frac{v^2}{2} + k^{\log_e \rho}\right)\right]$$

$$= -\frac{\partial}{\partial r}\left[v\frac{\partial v}{\partial t} + \frac{k}{\rho}\frac{\partial \rho}{\partial t}\right]$$

$$= -\frac{\partial}{\partial r}\left[v\left(-v\frac{\partial v}{\partial r} - \frac{k}{\rho}\frac{\partial \rho}{\partial r}\right) + \frac{k}{\rho}\left(-\frac{\partial (\rho v)}{\partial r}\right)\right] \text{ [using (i), (ii)]}$$

$$\therefore \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial r}\left[v^2\frac{\partial v}{\partial r} + \frac{vk}{\rho}\frac{\partial \rho}{\partial r} + \frac{kv}{\rho}\frac{\partial \rho}{\partial r} + k\frac{\partial v}{\partial r}\right]$$

$$= \frac{\partial}{\partial r}\left(v^2\frac{\partial v}{\partial r} + 2\frac{kv}{\rho}\frac{\partial \rho}{\partial r}\right) + k\frac{\partial^2 v}{\partial r^2}$$

$$= \frac{\partial}{\partial r}\left[v^2\frac{\partial v}{\partial r} + 2v\left(-\frac{\partial v}{\partial t} - v\frac{\partial v}{\partial r}\right)\right] + k\frac{\partial^2 v}{\partial r^2} \text{ [using (ii)]}$$

$$= \frac{\partial}{\partial r}\left[-v^2\frac{\partial v}{\partial r} - 2v\frac{\partial v}{\partial t}\right] + k\frac{\partial^2 v}{\partial r^2}$$

Hence, $$\frac{\partial^2 v}{\partial t^2} + \frac{\partial}{\partial r}\left(v^2\frac{\partial v}{\partial r} + 2v\frac{\partial v}{\partial t}\right) = k\frac{\partial^2 v}{\partial r^2}$$

Ex.-5. *A sphere is at rest in an infinite mass of homogeneous liquid of density ρ, the pressure at infinity being Π, show that, if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time is* $\Pi + \frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]$.

Ans. The equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0 - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

and the equation of continuity is

$$x^2 v = F(t)$$

i.e. $$\frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$$

Hence, $$\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$ as ρ is constant.

Int. w.r.t. 'x', $$-\frac{F'(t)}{x^2} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C_0 \quad \text{.............. (1)}$$

Boundary conditions are :

i) when $x \to \infty, p = \Pi, v = 0;$

ii) when $x = R, p = p, v = \dot{R}$

Also $x^2 v = F(t) = R^2 \dot{R}$

$$\therefore F'(t) = 2R\dot{R}^2 + R^2 \ddot{R}$$

Using (i) and (ii) in (1) we get

$$0 + 0 = -\frac{\Pi}{\rho} + C_0 \Rightarrow C_0 = \frac{\Pi}{\rho}$$

and $$-\frac{F'(t)}{R} + \frac{1}{2}\dot{R}^2 = -\frac{p}{\rho} + C_0 = -\frac{p}{\rho} + \frac{\Pi}{\rho}$$

$$\therefore \frac{p}{\rho} = \frac{\Pi}{\rho} - \frac{1}{2}\dot{R}^2 + \frac{1}{R}\left[2R\dot{R}^2 + R^2\ddot{R}\right]$$

or, $$p = \Pi + \frac{1}{2}\rho\left[3\dot{R}^2 + 2R\ddot{R}\right] \quad \text{................ (2)}$$

$$= \Pi + \frac{\rho}{2}\left[\dot{R}^2 + 2\dot{R}^2 + 2R\ddot{R}\right]$$

$$= \Pi + \frac{\rho}{2}\left[\dot{R}^2 + \frac{d}{dt}\left(2R\dot{R}\right)\right]$$

$$= \Pi + \frac{\rho}{2}\left[\dot{R}^2 + \frac{d}{dt}\left(\frac{dR^2}{dt}\right)\right]$$

$$\therefore p = \Pi + \frac{\rho}{2}\left[\dot{R}^2 + \frac{d^2R^2}{dt^2}\right].$$

Ex.6. *Liquid is contained between two parallel planes; the free surface is a circular cylinder of radius 'a' whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius 'b' is suddenly annihilated. Prove that if Π be the pressure at the outer surface, the initial pressure at any point of the liquid, distant r from the centre, is* $\Pi\left[\frac{\log r - \log b}{\log a - \log b}\right]$.

Ans. The equation of continuity is

$$xv = F(t)$$

and equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\therefore \frac{F'(t)}{x} + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{p}{\rho}\right)$$

Int., w.r.t. 'x', $F'(t)\log x + \frac{v^2}{2} = -\frac{p}{\rho} + C_0$ .... (1)

Since, initially the fluid is at rest, so we have the boundary conditions:

i) when $x = a, v = \dot{x} = 0, p = \Pi, t = 0$;

ii) when $x = b, v = \dot{x} = 0, p = 0, t = 0$;

iii) when $x = r, t = 0, p = p_0$ say.

Using (i) and (ii) in (1), we get

$$F'(0)\log_e a = -\frac{\Pi}{\rho} + C_0 \Rightarrow C_0 = \frac{\Pi}{\rho} + F'(0)\log_e a$$

and $\quad F'(0)\log_e b = 0 + C_0 \Rightarrow C_0 = F'(0)\log_e b$

which implies that

$$F'(0)\log_e a + \frac{\Pi}{\rho} = F'(0)\log_e b$$

i.e. $F'(0)\log_e\left(\frac{b}{a}\right)=\frac{\Pi}{\rho}$ .... (2)

Hence $C_0=F'(0)\log_e b=-\frac{\Pi}{\rho}\cdot\frac{\log_e b}{\log_e\left(\frac{a}{b}\right)}$ .... (3)

Again using (iii) in (1), we get

$$F'(0)\log_e r=-\frac{p_0}{\rho}+C_0$$

or, $$-\frac{\Pi}{\rho}\frac{\log r}{\log_e\left(\frac{a}{b}\right)}=-\frac{p_0}{\rho}-\frac{\Pi}{\rho}\frac{\log_e b}{\log_e\left(\frac{a}{b}\right)}\quad\text{[using (2), (3)]}$$

$$\therefore p_0=\Pi\left(\frac{(\log_e r-\log_e b)}{\log_e\left(\frac{a}{b}\right)}\right)=\Pi\left(\frac{(\log_e r-\log_e b)}{(\log_e a-\log_e b)}\right)$$

Ex.7. *A velocity field is given by* $\vec{V}=(x\vec{j}-y\vec{i})/(x^2+y^2)$. *Determine whether the flow is irrotational. Calculate the circulation round*

i) *a square with corners at* (1,0), (2,0), (2,1), (1,1);

ii) *on unit circle with centre at the origin.*

Ans. The flow of the fluid is to be irrotational if $\vec{\nabla}\times\vec{V}=\vec{0}$.

$$\text{Now } \vec{\nabla}\times\vec{V}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0\end{vmatrix}\quad\text{where } \vec{V}=\frac{x\vec{j}-y\vec{i}}{x^2+y^2}$$

$$=0.\vec{i}+0.\vec{j}+\vec{k}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]$$

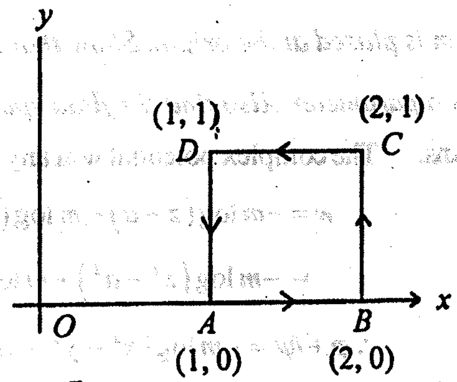
$$= \vec{k}\left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right]$$

$$= \vec{k}\left[\frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2}\right]$$

$$= \vec{k}\cdot 0$$

$$= \vec{0}$$

which shows that the fluid motion is irrotational.



i) The circulation of the flow around the closed path C is

$$\Gamma = \int_C \vec{V}\cdot d\vec{r}$$

$$= \int_S (\vec{\nabla}\times\vec{V})\cdot \vec{n}dS \text{ (by Stoke's theorem)}$$

Here $\vec{V}$ must be a continuous differentiable over S. At given problem $\vec{V}$ is not continuous at the origin but origin does not lie inside the rectangle so that Stok's theorem is applicable. But we have proved $\vec{\nabla}\times\vec{V} = \vec{0}$. Hence

$$\Gamma = 0$$

ii) In this case the equation of the path C is $x^2+y^2=1$ whose centre is at origin i.e. (0,0). Hence the circle C contains origin, the point of singularity. So, Stoke's theorem is not applicable.

Therefore, $\Gamma = \int_C \vec{V}\cdot d\vec{r} = \int_C \left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right)$

$$= \int_C \frac{-\left[\frac{ydx - xdy}{y^2}\right]}{1+\left(\frac{x}{y}\right)^2}$$

$$= \int_C \frac{-d\left(\frac{x}{y}\right)}{1+\left(\frac{x}{y}\right)^2} = -\left[\tan^{-1}\left(\frac{x}{y}\right)\right]_C$$

$$= -\tan^{-1}(\cot\theta)_C \text{ [where } x = r\cos\theta, y = r\sin\theta\text{]}$$

$$= -\left[\frac{\pi}{2}-\theta\right]_0^{2\pi} \text{ [for the whole circle, } \theta \text{ runs 0 to } 2\pi\text{]}$$

$$= 2\pi.$$

Ex.8 *Two sources, each of strength m, are placed at the points $(-a,0)$ and $(a,0)$ and a sink of strength 2m is placed at the origin. Show that the stream lines are curves $(x^2+y^2)^2 = a^2[x^2-y^2+\lambda xy]$ where λ is a parameter. Also find the fluid speed at any point.*

Ans. The complex potential w at any point $P(z)$ is given by

$$w = -m\log(z-a) - m\log(z+a) + 2m\log(z-0)$$

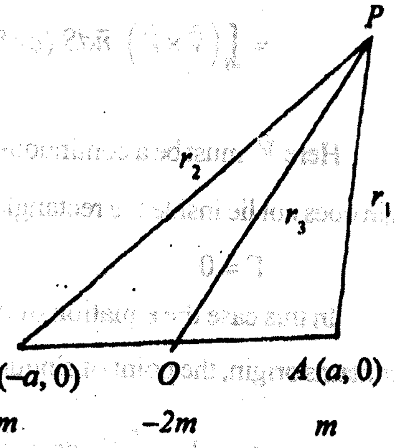
$$= -m\log(z^2-a^2) + m\log z^2$$

$$\therefore \varphi + i\psi = -m\log(x^2-y^2+2ixy-a^2)$$

$$+m\log(x^2-y^2+2ixy) (\because z = x+iy)$$

$$\Rightarrow \psi = -m\tan^{-1}\left(\frac{2xy}{x^2-a^2-y^2}\right) + m\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$$

$$= -m\tan^{-1}\left(\frac{2a^2xy}{(x^2-y^2)(x^2-a^2-y^2)+4x^2y^2}\right).$$



Since stream lines are given by

$$\psi = \text{constant}$$

or, $$-m\tan^{-1}\left(\frac{2a^2xy}{(x^2-y^2)(x^2-a^2-y^2)+4x^2y^2}\right) = m\tan^{-1}\left(\frac{2}{\lambda}\right), \text{ say}$$

or, $$\lambda a^2xy = (x^2-y^2) - a^2(x^2-y^2) + 4x^2y^2$$

or, $$\lambda a^2xy = (x^2+y^2)^2 - a^2(x^2-y^2)$$

or, $(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy)$, where λ is a variable parameter.

Now speed of the flow =

$$\left|\frac{dw}{dz}\right| = \left|-\frac{2mz}{z^2-a^2}+\frac{2mz}{z^2}\right| = \left|\frac{2ma^2}{z\left(z^2-a^2\right)}\right|$$

$$= \frac{2ma^2}{|z|\cdot|z-a|\cdot|z+a|} = \frac{2ma^2}{r_1r_2r_3}$$

Ex.9. *Parallel line sources (perpendicular to the xy-plane) of equal strength m are placed at the points $z = nia$, where $n = \ldots\ldots\ldots\ldots, -2,-1,0,1,2,\ldots$ prove that the complex potential is $w = -m\log\sinh\left(\frac{\pi z}{a}\right)$. Hence find the complex potential for two dimensional doublets, with their axes parallel to the x-axis, of strength μ at the same points, is given by $w = \mu\coth\left(\frac{\pi z}{a}\right)$.*

Ans. The complex potential w at any point z due to the system of sources of equal strength m placed at $z = \pm nia$ is given by

$$w = -m\log(z-0) - \sum_{n=1}^{\infty} m\log(z-nia) - \sum_{n=1}^{\infty} m\log(z+nia)$$

$$= -m\log z - \sum_{n=1}^{\infty} m\log\left(z^2+n^2a^2\right)$$

$$= -\sum_{n=1}^{\infty} m\log\left(1+\frac{z^2}{n^2a^2}\right)n^2a^2\cdot z$$

$$= -\sum_{n=1}^{\infty} m\log\left(1+\frac{z^2}{n^2a^2}\right)\cdot\frac{\pi z}{a} - \sum_{n=1}^{\infty} m\log\left(n^2a^2\cdot\frac{a}{\pi}\right)$$

$$= -\sum_{n=1}^{\infty} m\log\left(1+\frac{z^2}{n^2a^2}\right)\frac{\pi z}{a} \text{ (omitting constant)}$$

$$= -\sum_{n=1}^{\infty} m\log\theta\left(1+\frac{\theta^2}{n^2\pi^2}\right), \left[\text{put } \theta = \frac{\pi z}{a}.\right]$$

$$= -m\log\theta\left(1+\frac{\theta^2}{\pi^2}\right)\left(1+\frac{\theta^2}{2^2\pi^2}\right)\left(1+\frac{\theta^2}{3^2\pi^2}\right)\ldots\ldots$$

$$= -m\log\sinh\theta$$

$$= -m \log \sinh\left(\frac{\pi z}{a}\right).$$

If $w = -m\log(z-a)$, due to a source $+m$ at $z=a$ then $w = -\frac{m}{z-a}$, due to the doublet $+m$ at $z=a$ with its axis along x-axis, i.e., $w = \frac{d}{dz}\left[m\log(z-a)\right]$ for a doublet $+m$ at $z=a$ with its axis along x-axis.

So, complex potential for the doublets of strength m at these points is given by

$$w = -\frac{d}{dz}\left[m \log \sinh\left(\frac{\pi z}{a}\right)\right]$$

$$= \frac{m\pi}{a}\coth\left(\frac{\pi z}{a}\right)$$

$$\mu \coth\left(\frac{\pi z}{a}\right).$$

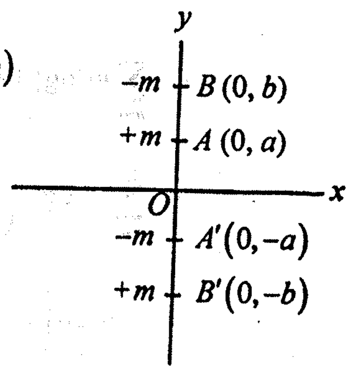
Ex 10. *If the fluid fills the region of space on the positive side of x-axis, which is a rigid boundary, and if there be a source +m at the point $(0,a)$, and an equal sink at $(0,b)$, and if the pressure on the negative side of the boundary be the same as the pressure of the fluid at infinity, show that the resultant pressure on the boundary is $\pi\rho m'(a-b)^2/ab(a+b), \rho$ is the density of the fluid.*

Ans. Here object system consists of source $+m$ at $A(0,a)$ i.e., at $z = ia$ and sink $-m$ at $B(0,b)$ i.e., at $z = ib$. So, image system consists of source $+m$ at $A'(0,-a)$ i.e., at $z = -ia$ and sink $-m$ at $B'(0,-b)$ i.e., at $z = -ib$ w.r.t. the positive line Ox which is a rigid boundary.

The complex potential due to object system with rigid boundary is equivalent to the object system and its image system with no rigid boundary. Hence the complex potential is

$$w = -m\log(z-ia) + m\log(z-ib) - m\log(z+ia) + m\log(z+ib)$$

$$= -m\log(z^2+a^2) + m\log(z^2+b^2)$$



$$\therefore \frac{dw}{dz} = -2mz\left[\frac{1}{z^2+a^2} - \frac{1}{z^2+b^2}\right] = \frac{2mz(a^2-b^2)}{(z^2+a^2)(z^2+b^2)}.$$

Hence speed of the flow is

$$|\vec{V}| = \left|\frac{dw}{dz}\right| = \frac{2m\left(a^2 - b^2\right)|z|}{\left|z^2 + a^2\right| \cdot \left|z^2 + b^2\right|}$$

So the speed of the flow at any point on x-axis is given by replacing z by x. So,

$$|\vec{V}|_x = \frac{2mx\left(a^2 - b^2\right)}{\left(x^2 + a^2\right)\left(x^2 + b^2\right)} \quad \text{..............(i)}$$

Bernouli's equation for steady motion is

$$\frac{p}{\rho} + \frac{1}{2}|\vec{V}|^2 = C$$

Let $p = p_0, |\vec{V}| = 0$ when $x \to \infty$

$$\therefore C = \frac{p_0}{\rho}$$

Here $\frac{1}{2}|\vec{V}|^2 = \frac{p_0 - p}{\rho}$ ....(ii)

Let p' be the required pressure on the boundary which is given by

$$p' = \int_{\infty}^{\infty}(p_0 - p)\,dx = \int_{\infty}^{\infty}\frac{1}{2}|\vec{V}|^2 \rho dx \text{ [using (ii)]}$$

$$= \frac{1}{2}\rho\int_{-\infty}^{\infty}\frac{4m^2x^2\left(a^2 - b^2\right)^2}{\left(x^2 + a^2\right)^2\left(x^2 + b^2\right)^2}dx$$

$$= 4m^2\rho\left(a^2 - b^2\right)^2 \int_0^{\infty}\frac{x^2 dx}{\left(x^2 + a^2\right)^2\left(x^2 + b^2\right)^2}$$

$$= 4m^2\rho\left[\left(\frac{a^2 + b^2}{a^2 - b^2}\right)\left(\frac{\pi}{2b} - \frac{\pi}{2a}\right) - \frac{\pi}{4a} - \frac{\pi}{4b}\right]$$

$$\left[\because \int_0^{\infty}\frac{dx}{x^2 + a^2} = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^{a} = \frac{\pi}{2a}\right.$$

and $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{1}{a^3}\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$, put $x = a\tan\theta$

$$= \frac{\pi}{4a^3}\Bigg]$$

$$= \frac{\pi\rho m^2(a-b)^2}{ab(a+b)}.$$

6.31 Self Assessment Questions:

1. Water oscillates in a bent tube (uniform) in a vertical plane. If O be the lowest point of the tube, AB the equilibrium level of water, α and β the inclinations of the tube to the horizontal at A, B and $OA=a, OB=b$, the period of oscillation is given by $2\pi\left[(a+b)/g(\sin\alpha+\sin\beta)\right]^{1/2}$.

2. An infinite mass of homogeneous incompressible fluid is at rest subject to a uniform pressure Π and contains a spherical cavity of radius a, filled with a gas at a pressure $m\Pi$; prove that if the gas obeys Boyle's law, the radius of the sphere will oscillate between the values a and na, where n is given by $1+3m\log n - n^3 = 0$.

3. A mass of liquid, of density ρ and volume $\frac{4}{3}\pi c^3$, is in the form of a spherical shell; a constant pressure Π is exerted on the external surface of the shell; there is no pressure on the internal surface and no other forces act on the liquid; initially the liquid is at rest and the internal radius of the shell is $2c$; prove that the velocity of the internal surface, when its radius is c, is $\left[\frac{14\pi}{3\rho}\cdot\frac{2^{1/3}}{2^{1/3}-1}\right]^{1/2}$.

4. If $\tilde{w}$ is the impulsive pressure; φ and φ' the velocity potentials immediately before and after an impulse acts, V the potential of impulses, prove that $\tilde{w}+\rho V+\rho(\varphi-\varphi') = \text{constant}$, ρ is the density of the fluid.

5. Between the fixed boundaries $\theta = \frac{\pi}{6}$ and $\theta = -\frac{\pi}{6}$, there is a two dimensional liquid motion due to a source at the point $(r=c, \theta=\alpha)$ and a sink at the origin, absorbing water at the same rate as the source produces it. Find the stream function, and show that one of the stream lines is a part of the curve $r^3\sin 3\alpha = c^3\sin 3\theta$.

6. Prove that $u = -wy, v = wx, w = 0$ represents a possible motion of inviscid fluid. Find the stream function and sketch stream lines. What is the basic difference between that motion and one represented by the potential $\varphi = A \log_e r, r = \sqrt{x^2 + y^2}$.

7. If λ be a variable parameter, and f a given function, find the condition that $f(x, y, \lambda) = 0$ should be a possible system of stream lines for steady irrotational motion in two dimentions.

8. Use the method of images to prove that if there be a source m at $z = z_o$ in the fluid bounded by the lines $\theta = 0, \theta = \pi/3$, the solution is $\varphi + i\psi = -m \times \log(z^3 - z_0^3)(z^3 - z_0'^3)$ where $z_0 = x_0 + iy_0, z_0' = x_0 - iy_o$.

6.32 Further Suggested Readings

1. Continuum Mechanics: T.J. Chung, Prentice-Hall.
2. Schaum's outline of theory and problem of continuum mechanics: Gedrge R. Mase, McGraw-Hill.
3. Continuum Mechanics: A.J.M. Spencer, Longman.
4. Mathematical Theory of Continuum Mechanics: R.N. Chatterjee, Narosa Publishing House.
5. Foundation of fluid mechanics: S.W. Yuan, Prentice-Hall.
6. Fluid Dynamics: J.K. Goyal, K.P. Gupta, Pragati Prakashan.
7. Textbook of Fluid Dynamics: F. Chorlton, CBS Publishers and Distributors.
8. Theory of Elasticity : Yu. Amenzade, Mir Publishers, Moscow.
9. Applied Elasticity : C.T. Wang, McGraw-Hill.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-V

Group-B

Module No. - 55
DOS & WINDOWS 95/98

Content :

1.0 Introduction

The Graphical User Interfeace (GUI) is one of the most revolutionary changes to occur in the evolution of modern computing system. In less than 10 years, the expectation of what the interaction between human and computer would be like was changed from a terse, character-oriented system to the now more graphics-iriented system. This revolution has increased the aceessibility and usuability of computer systems to the general public. In this unit, the software MS-Windows, the most popular GUI for personal computers is discussed. Also the command based software, MS-DOS is discussed shortly.

2.0 Objectives

In this unit, all basic operations such as copy, move a file, to create a folder, to delete a folder, to rename a file and folder, to get the file information, to increase the memory size, to back up utility, to check all drives, to access accessoreis, etc. in Windows 95 are discussed.

3.0 Key Words and Study guides

Copy, create, delete, folder, rename, back up, accessories.

4.0 Main Discussion

4.1 MS-DOS Basics

The Command Prompt : When you first turn on your computer, you will se some information flash by. When the information stops scrolling past, you will se the following :

C:\>

This is called the **command prompt.** The flashing underscore next to the command prompt is called the **cursor.** The cursor shows where the command you type will appear.

Typing a command : After typing a command, you must press ENTER. For example, to type a command at the command promt, do the following :

- Type the following command at the command prompt : **ver**
- Press ENTER

Viewing the contents of a Directory : To view the contents of a directory,

* Type the following at the command prompt : **dir**
* Press ENTER

Some information will be shown. this is called a **directory list.** A directory list is a list of all files and sub directories that a directory contains.

To view the contents of a directory one screen at a time,

* Type the following at the command prompt, **dir/p**
* Press ENTER

One screen of information appears. At the bottom of the screen, you will see the following message : *Press any key to continue.*

* To view the next screen of information, press any key on your keyboard.

Repeat this step until the command prompt appears.

To view the contents of a directory in wide format,

* Type the following at the command prompt : **dir/w**
* Press ENTER

The directory list appears, with the filenames listed in wide format. Only filenames are listed. No information about the files' size or date and time of creation appears.

* If the directory contains more files than wil fit on one screen, you can combine the **/p** and **/w** switches as follows: **dir/w/p**

Chanding Directories : In a directory list, all the names that have <DIR> beside them are directories. To change from the root directory to the another directory (DOS, saying),

* Type the following at the command prompt : **cd dos**
* Press ENTER

The command prompt changes. It should now look like the following :

C:\DOS>

The command prompt shows which directory you are in. In this case, you know that you successfully changed to the DOS directory because the command prompt displays the directory's name. Now the current directory is DOS. The **cd** command stands for **"change directory"**.

Changing Back to the Root Directory : To change to the root directory do the following :

- Type the following at the command prompt : **cd**
- Pres ENTER

We note that the slash typed in this command prompt is a backslash (\), not a forward slash (/). No matter which directory you are in, this command always returns you to the root directory of a drive. The root directory does not have a name. It is simply referred to by a backslash (\).

Creating a Directory : To create a director, , ou will use the **md** command. The **md** command stnads for "make directory". To create and change to a directory named VU,

- Type the following at the command prompt : **md/VU**
- Press ENTER

You have now created a directory named VU.

- To change to the new VU directory, type the following at the command prompt : **cd\VU.** You now create a directory within the VU directory, named MATH as follows:
- Type the following at the commands prompt : **md MATH**
- Press ENTER
- To change to the MATH directory, type the following ~~at the~~ command prompt : **cd MATH**
- Press ENTER

The command prompt should now look like : C:\VU\MATH>

- To switch back to the VU directory, type the following : **cd...**
- Press ENTER

Deleting a Directory : To delete a directory, use the **rd** command. The rd command stands for **"remove directory"**. To delete a directory MATH, do the following:

* Make sure the command prompt looks like : C:\VU>
* Type the following at the command prompt : **rd MATH**
* Press ENTER

You cannot delete a directory if you are in it.

Changing Drives : To change to and view files on a different drive (Floppy drive, saying)

* Insert Floppy disk.
* Type the following at the command prompt: **a:**
* Press ENTER

The command prompt changed to the following : A:\>

* To view a list of files on the floppy disk in drive A, type at the command prompt : **dir**
* To change back to drive C, type the following at the command prompt : **c:**
* Press ENTER

The command prompt changed to the following : C:\>

To view files on drive A when drive C is current :

* Type the following at the command prompt : **dir a:**
* Press ENTER

Copying Files : To copy a file, you will use the **copy** command. When you use the **copy** command, you must include two parameters : the location and name of the file you want to copy, or the **source,** and the location to which you want to copy the file, or the **destination.** You separate the source and destination with a space. The **copy** command follows this pattern : *copy source destination.* To copy TEXT.DOC file from MATH directory to the VU directory :

* Return to root directory by typing the following at the command prompt :
  cd
* To copy the TEXT.DOC file, type the following at the command prompt :
  C:\>copy C:\VU\MATH\TEXT.DOC C:\VU
* Press RETURN

Or

* Go to the directory C:\VU\MATH>

* To copy the TEXT.DOC file, type the following at the command prompt:

  C:\VU\MATH>copy TEXT DOC. C:\VU

Press RETURN

The following message appears : 1 file (s) copied.

Copying a group of files, wildcards can be used. The asterisk (*) wildcard matches any character in that position and all other positions that follows it. For example, to copy the files with a TEXT extension to the VU directory, type the following at the command prompt : *copy*.txt C :\VU*

Renaming Files : To rename a file, you will use the **ren** command. The **ren** command stands for "rename". When you use the **ren** command, you must include two parameters. The first is the file you want to rename, and the second is the new name for the file. You separate the two names with a space. The **ren** command follows the following pattern :

ren oldname newname

For example, to rename the file TEXT.DOC to MKS. DOC in the directory *C:\VU>*

* Type the following at the command prompt : **ren TEXT.DOC MKS.DOC**
* Press ENTER

Deleting Files : To delete a file, you will use the **del** command. The **del** command stands for "delete".

To delete TEXT.DOC file

* Delete the TEXT.DOC file by typing the following at the command prompt :

  del TEXT.DOC

* To confirm that you deleted the file sucessfully, type the following at the commannd prompt :

  dir

Deleting Folders : To delete the MATH directory

* Return to the root directory by typing the following at the command prompt.

  cd
* To see the directory : **dir**
* Remove the MATH directory : **rd MATH**
* To veryfy that the MATH directory has been removed : **dir**

The MATH directory should not appear in the directory list.

Fomratting a Floppy Disk : When you purchase new floppy disks, you must generally format them before you can use them. You must remember that any data already in the disk will be ereased, if you format the disk. To format a floppy disk

* Type the following at the command prompt: **format a:**
* Press ENTER

The following message appears : Insert new diskette for drive A:

and press ENTER when ready:

* Insert the disk you want to format in drive A. When you are ready, press ENTER

When the format is complete, the following message appears : Volume label (11 characters, ENTER for none)?

* If you have another disk to format, press Y. If not, press N.

4.1.1. Freeing Disk space

There are two processes, by which the more disk space can be obtained : (i) deleting unnecessary files, and (ii) using scan disk.

Deleting unnecessary files : To delete unnecessary files, we use the **del** command. We use the following guidelines to decide which files to delete:

* Delete any temporary files created by your programs. Some programs store temporary files in a directory specified by the TEMP environment variable. To determine whether the computer has a directory designated for temporary files, we type **set** at the command prompt, and then check the value that MSDOS displays for the TEMP variable. periodically delete any files in the directory specified by the TEMP environment variable.

- If you have not used a file in a long time, we consider copying it to a floppy disk and then deleting it from the hard disk.

Using scandisk to free disk space : We can use the **scandisk** command to recover lost clusters that are taking up space on our hard disk. A cluster is a unit of storage that can get lost when a program unexpectedly stops running without saving or deleting temporary files properly. A lost cluster is marked as being in use, but it actually contains no useful data. Over time, lost clusters can accumulate and take up disk space. When scandisk finds lost clusters, it prompts you to specify whether we want save them. If we choose to save them, scandisk converts them to files that have a .CHK extension. We can then examine the contents of these .CHK files and delete them if they contain information we don't need. To recover lost clusters, do the followings:

- Quit all running programs.
- Change to the hard disk you want to check.
- Type **scandisk** at the command prompt, and press ENTER. Then type Y at the prompt.
- The scandisk program checks the drive for problems, including lost clusters. If scandisk finds lost clusters, it prompts you to specify whether you want to save them. To save the information the lost clusters, choose Save. The scandisk program converts any lost clusters to files with filenames such as FILE0000.CHK. It stores these files in your root directory.
- To examine the contents of .CHK file, use **more** command. Delete any .CHK files you do not want using the **del** command.
- Delete any .CHK files you do not want by using the **del** command. For example, to delete the file FILE000.CHK file, type the following at the command prompt: *del file*0000.*chk*

4.2 WINDOWS 95/98

4.2.1. Introduction

Microsoft Windows is the very popular Graphical User Interface (GUI) for personal computers. The GUI has the following common features:

- Secondary user-input devices. Usually a pointing device and typically a mouse.

- Point and shoot functionality with *screen menus* that appear or disappear under pointing-device-control.
- Windows that graphically display what the computer is doing.
- Icons that represent files, directories and other application and system entities.
- Dialog boxes, button, sliders, check boxes and many other graphical metaphors.

The major benefits of Windows are :

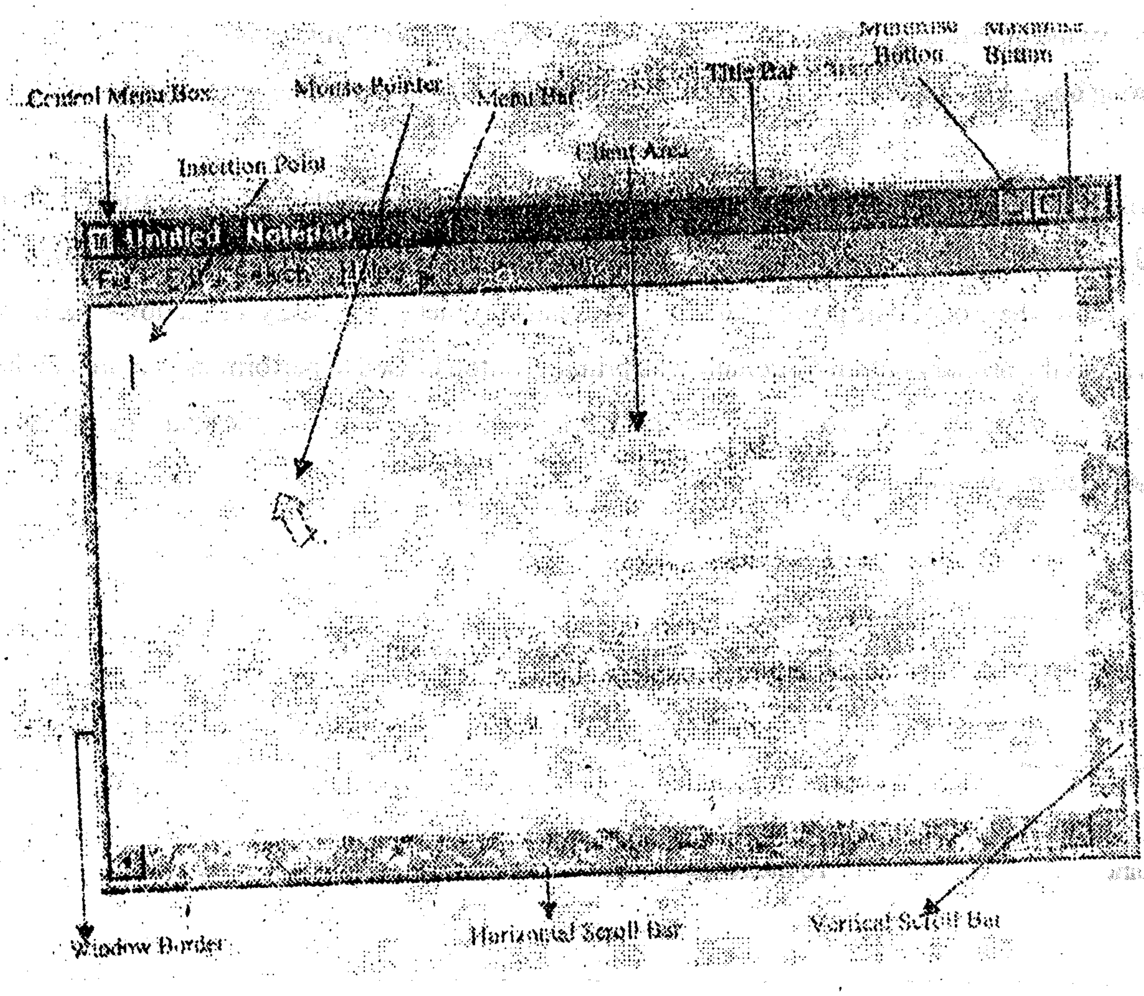
- Multi-tasking : Windows provides non-pre-emptive multi-tasking support, users can have several application in progress at the same time. Each application can be active in a separate window.
- Commson look : All windows applications have the same basic look and feel.
- Data Scharing : Windows allows data transfer between application clipboard. Any type of data can be transferred from one window with the clipboard, etc.

4.2.2. Structure of a Window

Depending upon the type of application all windoes may not have every element. Some parts, such as title bar and menu bar, are common for most of the windows. The following is the standard window. In a window there are different parts which are discussed below:

- Control menu box. It is in the upper left corner of each window. Clicking on the Control menu box opens the Control menu. The menu contains the options by which you can move, size, and close a window while working with the keyboard.
- Title bar : It displays the name of the application, document, or a folder name.
- Menu bar : It lists the available all menus which contain a list of commands.
- Scroll bars : There are two types of scroll bar : horizontal and vertical. These are used to move through a document when the entire document does not fit in a window.
- Minimize button : When you click on this button, it reduces the window to an icron and arranges it on the desktop. Minimizing the application window does not quit the application.
- Maximize button : When you click on this button, it enlarges the active window so that it fits into the entire desktop.

- Restore button : Clicking on this button returns the window to its previous size if it is maximized or minimized.
- Window border : It is the outside edge of the window. The window can be resized by lengthening or shortening the border.
- Insertion point : It is a flashing vertical bar that marks the place where text or graphics are to appear on typing or drawing.
- Mouse pointer : An arrow used for pointing items. It appears if the mouse is installed on the system.
- Client area : It is the area inside the window which is under the application control.



4.2.3 Some common tools

Icon : It is used to indicate a symbolic representation of any system such as file, folder, address, books, and applications and so on. A specific type of icon represents different types of objects. For example, a folder icon contains a group of files or other folder icons. Double clicking on the folder icon causes a window to be opened displaying a list of icons and folder icon.

Menu : It displays a list of commands available within a application. From the menu, the end-user can select different operations such as File, Edit, View, Window, Help etc. Instead of remembering commands at each stage, a menu can be used to provide a list of items. Menu item can be invoked by moving the cursor on the menu item and selecting the item by clicking the mouse. When a menu item is accessed it could cause other commands called the pull-down menu. Pull down menu is used to present a group of related commands.

Dialog box : A dialog box is used to collect the information from the user or to present information to the user.

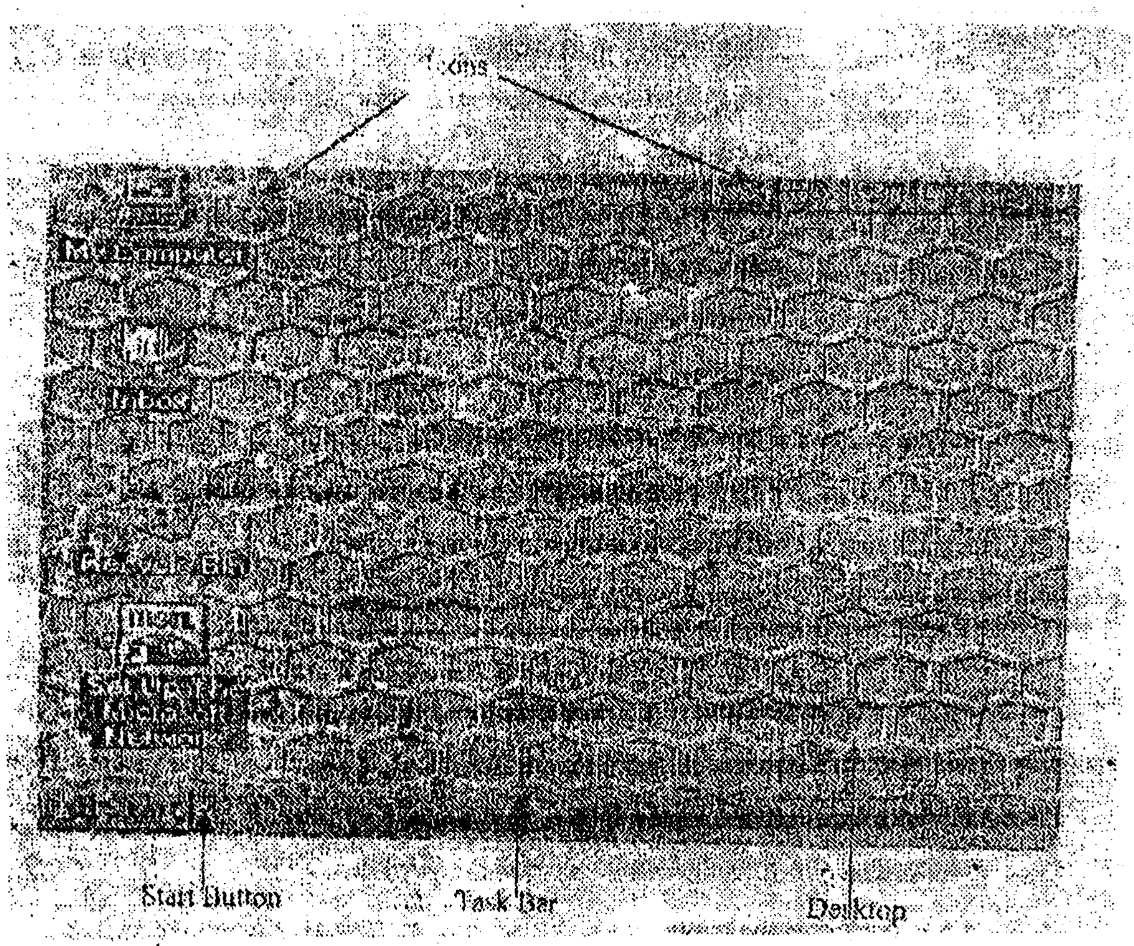
Basic Techniques for working in Windows : Using a mouse is usually easier and faster than using the keyboard but one needs to know both mouse and keyboard techniques to work in windoes. Generally the mouse has two buttons where one is the primary mouse botton and the other is secondary. In windows, the left mouse button is set as the primary button by default. The primary button is used to perform the majority of the tasks whereas the secondary button is to display shortcut menu for the current window application. The following table shows the functions of a mouse.

Term	Meaning
Click	To quickly press and release the primary mouse button to select a single item.
Double-click	To click the primary moust button twice in rapid succession to carry out an action after the item is selected
Point	To move the mouse until the mouse pointer on the screen points to the item of choice

Drag	To press and hold down the primary mouse button while moving the mouse
Drag and drop	Pointing to the item of choice, press and hold down the primary mouse button while moving the mouse. Release the mouse button on reaching the desired location to place the item

4.2.4. Starting Windows 95/98

When you switch on the computer, after booting you will come directly to Windoes 95 screen, which is called the desktop (metaphor) whose appearance is as follows:



When you click the right mouse button on the desktop, it gives the properties by which you can adjust settings such as desktop color and background, screen saver etc. On the desktop, there are different items discussed below:

Task bar : The bar that is generally at the bottom of the first screen i.e., the desktop is called the taks bar. It contains the *Start* button, which is used to quickly start different actions such as to find a file, to invoke an application etc. By default, the task bar and the Start menu are always visible shen Windoes is running. To customize the task bar, right click on it, and then click on the *Properties.*

Start Menu : When you click on the *Start* menu, then a popped-up menu will be displayed on the desktop. This contains different options which are as follows:

Option	Function
Programs	It displays a list of programs.
Documents	It displays a list of documents that you have opened previously.
Settings	It displays a list of system components for which you can change settings
Find	It enables you to find folder, file, shared computer, or mail message
Help	It starts help about Windows 95
Run	It starts a program or opens a folder when you type an MSDOS command
Shut down	It shuts down or restarts your computer.

When you click on the option that has a right pointed arrow, a cascading menu appears. If you want add or remove programs to and from the Start menu, then follow the following steps:

* Click the *Start* button and the choose the *Settings* option
* Click on the *Taskbar*

  The Taskbar dialog box is displayed.
* Click on the Start Menu programs tab

* Click on Add button and then click *Browse* to locate the program you want to add
* Double click to choose it
* Click Next and then double click the menu on which you want the program to appear
* Type the name that you want to see on the menu
* Click on the Finish button
* If Windows prompts you to choose an icon, click one and then click the Finish.

Set-up Screen Saver : It is set to save the wear and tear of the screen. When your computer is idle for the number of minutes specified in the Wait box, then the screen saver starts to action. After it has started if you want to remove this then move your mouse or press any key on the key board. The following steps are required to set up a screen saver :

* Right click on the desktop

  A pulled-down menu is displayed.
* Choose the option Properties from the pulled-down menu

  The display properties dialog box is displayed.
* Click on the Screen Saver tab
* Click on the down arrow under Screen Saver field and browse through the different screen savers
* Choose the one that you like
* Click on Settings button to customize the way it works
* You can set the password by choosing the option Password Protected
* Click on the Set Password button when it is activated and specify the password
* Set the timings
* Click on the OK button.

Shortcut Menu : It is the menu that appears, when the right clicks on any item, shows the most frequently used commands for that item. To use shortcut menus, Right-click a file or folder. You can also right-click an empty space on the taskbar or desktop. To display a shortcut menu that lists every available command for a file or folder, point to the file or folder, and then press SHIFT while right-clicking it.

Shutdown Windoes 95 : To shut down your computer properly, do the following:

* Click Start button
* Choose Shut Down from the popped-up menu
* Then click Shut down.

Important : do not turn off your computer until a message appears telling you that it is safe to do so.

4.2.5 Managing Windows 95/98

Windows 95 provides a lot of utilities for managing your system. These utilities include like disk utilities to maintain disks and access the maximum available space on the disk. The different utilities by which you can manage Windows 95 are discussed below.

4.2.5.1 My Computer : It is the place where the things you have on computer like programs, documents; data files are accessible in Windoes 95. If you want see what is on your computer, then the following steps are required.

* Double click on my *My Computer* icon

  A window containing differenct items is opened
* Double click on the icon for the drive you want to look at

  Windows displays the files and folders on the drive. Folders can contain files, programs and even other folders.
* To open a file or folder, double click. it.

Action of right click on a drive, control panel and printer : When an object among the drives, control panel, and printer, is selected, the different options are displayed which are as follows:-

Option	Function
Open	It opens the selected object in a window
Explore	It opens the explorer to see the file structure of the drive

Find	It helps you search for files on the selected drive or anywhere on the filling system
Sharing	It lets you share the selected drive with other users in your organization that are connected on your computer
Format	It erases or formates the selected drive
Create Shortcut	It places a shortcut for the object on Windows 95 desktop for quick access.
Properties	It displays the information about the selected object.
Close	It closes the open window.

Creating a shortcut : A shortcut is a quick way to start a program or open a file or folder without having to go to its permanent location in Windows Explorer. Shortcuts are especially useful for programs, files, and folders you use frequently. There are three ways you can create a shortcut:

To create a shortcut in a folder :

1. In **My Computer** or Windows Explorer, click the folder in which you want to create the shortcut.
2. On the **File** menu, point to **New**, and then click **Shortcut.**
3. Follow the instructions on the screen.

To put a shortcut on the desktop

1. In **My Computer** or in the right pane of Windows Explorer, click the item, such as a file, program, folder, printer, or computer, for which you want to create a shortcut.
2. On the **File** menu, click **Create Shortcut.**
3. Drag the shortcut icon onto the desktop.

Notes

- To change any settings for the shortcut, such as what kind of window it starts in, right-click the shortcut, and then click **Properties.**
- To delete a shortcut, drag it to the **Recycle Bin.** The original item still exists on the disk.

4.2.5.2. System settings : The system can be personalized by making interesting and useful changes to Windows 95 settings. There are many settings you can make to make your workplace more interesting. Some of these are mentioned below:

* Change background of your desktop
* Adjust the double-click speed for your mouse
* Change the capacity of the Recycle Bin
* Change number, currency, time and date settings
* Change printer settings
* Change the number of colors your monitor displays
* Change settings for network service
* Change the screen resolution
* Have your monitor automatically turn off
* Change display fonts
* Protect your screen by setting up a screen saver
* Configure multimedia devices
* View or change resource settings for a hardware device
* Enable multiple users to personalize settings

All these are explained below. You can start the setting of any choice with the following steps:

* Click on the Start menu on the desktop
* Choose the Settings option from the popped-up menu
  A cascading menu is displayed.
* Choose the one you want to do

4.2.5.2.1. Control Panel : When you click on the option ***Control Panel*** from the cascading menu of ***Settings,*** a window is opened that contains the icon of the utilities for changing hardware configuration or customizing the Windows 95. You can also access the utilities of Control Panel, if you click on My Computer and then on Control Panel icon. The icon of utilities are Accessibility options, Add New Hardware, Add/Remove Programs. Date/

Time, Display, Fonts, Joystick, Keyboard, Modems, Mouse, Multimedia, Network, ODBC, Passwords, Printers, Regional Settings, Sounds, System whose setting are discussed as follows:

Date and Time : To update date and time settings of your system, follow the steps given below:

* Select ***Date/Time*** icon from the displayed icons and double click on it

  The Date and Time properties dialog box is displayed.
* Click the down-arrow in the month field to choose the correct month
* Click the up or down arrow in the year field to choose the current year
* Click the current day in the calender
* Set the new time field by clicking the up or down arrow or click anywhere in the text box and type a new number
* To set the correct time zone, click the Time Zone tab and then click your current location on the map of the world that is displayed.
* Click on the OK button.

Add New Hardware : To set up a new hardware, do the followings :

* Choice the ***Add New Hardware*** icon
* Double click on it

  The Add New Hardware wizard is displayed. It is recommended that you let Windows detect your hardware or installed its components in your computer before running the wizard.
* Follow the instruction on your screen.

Fonts : In Windows 95, there is a collection of Fonts. The fonts can be viewed, added, or removed from the available font list, which is discussed below.

To view fonts on your computer, the following steps are required:

* Double click the ***Fonts*** icon to open the Fonts Folder
* Double click the icon for the font

To add a new font to your computer, do the followings:

- Open the Fonts Folder
- On the File menu, click Install New Font
- Click the drive and folder that contain the fonts you want to add
- Double clickthe icon for the font you want to add

  To select more than one font to add, press and hold down the CTRL key, and then click the fonts you want. To select a range of fonts in the list, press and hold down the SHIFT key while dragging the cursor over the fonts.

- Open the Fonts folder
- Click the icon for the font you want to delete
- Click delete on the File menu

  To select more than one font to delete, press and hold down the CTRL key, and then click the fonts you want.

Look of Windows : It includes wallpaper and pattern in the background of the desktop, cursor blink rate or item size, color and fonts, which are discussed as follow:

The steps to change the background of your desktop are as follows:

- Choose the Display icon and double click it

  The Display propertied dialog box is displayed.

- In the Pattern or wallpaper list, click the pattern or wallpaper you want to use
- To cover your entire screen with a small wallpaper image, click Title
- To center a wallpaper image, click center
- Click on the Apply button and then OK.

The steps to change the way the items on your desktop look are as follows:

- If you want to change the appearnace of only one screen element, click that element in the item list. Then change the settings in the Item Size and Color and Font Size and Color areas.
- If you want to change the appearance of all screen elements simultaneously, click an appearance scheme in the Scheme list.

The steps to use larger or smaller display fonts are as follows:

- In the Font Size box, click the size you want your displayed fonts to be
- To customize the size of displayed fonts, click Custom
- If the Font Size area is unavailable, make sure your Desktop Area setting is higher that 640 by 480 pixels. If 640 by 480 pixels is the only setting available to you, you cannot change your display font.

To adjust the rate at which your cursor blinks, drag the slider in the Cursor Blink Rate area.

Keyboard : Keyboard layout affects which characters appear when you press the keys on your keyboard.

The steps to change the keyboard layout are given below:

- Choose the Keyboard icon

  The Keyboard Properties dialog box is displayed.
- Click the language tab
- Click the Properties button
- Select a different keyboard layout.
- Click OK button.

The steps to change the way your keyboard responds are as follows:

- Click the Speed tab after clicking the Keyboard icon
- If you want to adjust how much time elapses before a held down key begins repeating, drga the Repeat Delay slider
- If you want to adjust how quickly characters repeat when you hold down a key, drag the Repeat Rate slider
- Click OK button.

Mouse : The settings for the mouse include the button configuration, double click speed of the mouse pointer speed and pointer trail, mouse pointer shapes, and switching over to different mouse driver. To change the settings of mouse, follow the steps below:

- Choose the Mouse icon after clicking Control Panel

The Mouse Properties dialog box is displayed. It contains three tabs-Buttons, Pointers and Motion. These tabs are for different purpose of settings.

* Choosethe one that you wish to change the properties

The steps to reverse your mouse buttons are as follows:

* Click the Buttons tab from the Mouze Properties dialog box
* Click the Right-Handed or Left-Handed in the Button Configuration
* Click on the OK button.

The steps to adjust the double-click speed for your mouse are as follows:

* Click the Buttons tab from the Mouze Properties dialog box
* Drag the slider in the Double-Click Speed area
* To test the speed, double-click the image in the Test area
* Click on the OK button.

The steps to change the appearance of your mouse pointer are as follows:

* Click on the Pointers tab from the Mouse Properties dialog box
* To change all your pointers at one time, select a different scheme in the Scheme list
* To change only the pointer, click it, click Browse and then double-click the filename of the pointer you want to use
* Click on the OK button.

The steps to adjust the speed of your mouse pointer are as follows:

* Click the Motion tab from the Mouze Properties dialog box
* Drag the slider in the Pointer Speed area
* Click on the OK button.

  Changing the speed of your mouse pointer causes the pointer to respond more quickly or slowly to the movements of the mouse itself.

To turn on and adjust the mouse pointer trail, the following steps are :

* Click the Motion tab from the Mouze Properties dialog box
* In the Pointer Trail area, make sure the Show Pointer Trails box is checked

* To adjust the length of the pointer trail, drag the slider
* Click on the OK button.

Regional Settings : While installing Windows 95, you have to select a country, on which the number formats, currency, time and date formats are depended. These formats are used in various windows programs. To change the settings, the steps are as follows:

* Select the Regional Settings icon from the Control Panel

  The Regional Setting Properties dialog box is displayed.
* Select the formats appropriae for your country
* Click on the Regional Settings tab
* On the map displayed, click the region and Windows 95 changes its formats to match that region.
* For special adjustments, click the Number, Currency, Time, or Date Tab and change settings
* Click on the OK button.

4.2.5.2.2 Printers : The steps to change printer settings are as follows:

* Select the Printers icon from the Control Panel and double click on it
* Click the icon for the printer you are using from the Printers window
* Click Properties on the File menu

  The settings you can change depend on the type of printer you have
* Click the different tabs to see all of the options you can set.

The steps to set up a new printer do the following:

* Click the Start menu

  The popped up menu is displayed.
* Choose the Setting option
* Choose the Printer option
* Select Add Printer icon from the Printers window
* Follow the instruction when Add Printer Wizard is opened.
* If you want to print a test page, first make sure your printer is on and ready to print.

4.2.5.3 Back up : You can use Backup to back up files on your hard disk. You can back up files to floppy disks, a tape drive, or another computer on your network. If your original files are damaged or lost, you can restore them from the backup.

The steps to start Backup follow the followings :

* Click **Start** menu, then pointing to **Programs**
* Point to **Accessories**
* Point to **System Tools**
* Click the **Backup.**

If you do not see Backup on the Accessories menu, it is not installed. To install the Backup utility, follow the instructions given below:

* Start the Add/Remove Programs utility from the Control Panel window
* Click the Windows Setup tab
* Click the Disk Tools option in the window
* Then click OK to install the Backup utility, and then follow the instructions on the screen.

To add or remove a Windows component

* To open the **Add/Remove Programs Properties** dialog box at the **Windows Setup** tab.
* Under **Components,** click the component you want to add or remove.
  * To add all parts of the component, select its check box.
  * To remove all parts of the component, click to clear its check box.
  * To add or remove some parts of the component, click **Details,** and then click to select or clear check boxes for those parts.

Notes

* You can also open the **Add/Remove Programs Properties** dialog box at the **Windows Setup** tab by clicking **Start,** pointing to **Settings,** clicking **Control Panel,** double-clicking **Add/Remove Programs,** and then clicking the **Windows Setup** tab.
* If you used a compact disc to install Windows, you will be prompted to insert it into your computer.

4.2.6. Disk drive utilities

Windows 95 includes utilities to format diskettes, check diskettes and hard drives for errors, defragment disks and so on. Some of these utilities are discussed below. To access these utilities, the following steps are follows:

* Click the Start menu and then select Programs
* Choose Accessories
* Then Choose the System Tools

  The cascading menu that appears includes the various Disk Drive utilities.

4.2.6.1. Disk Defragmenter : It is used to rearrange files and unused space on the hard disk so that programs run faster. To speed up your hard disk by using Disk Defragmenter, follow the instructions given below:

* Select Disk Defragmenter from the cascading menu of System Tools

  Select Drive dialog box is appeared.
* Select the drive you want to defragmenter
* Click the drive you want to defragment
* Click the OK button
* If you want to change the settings that Disk Defragmenter uses, click Advanced option.
* Click Start button.

  While Windows degragments the selected disk, the computer can safely carry out other tasks. However, the computer will operate more slowly. To temporarily stop Disk Defragmenter so you can run other programs at full speed, click the Pause button.

4.2.6.2 Increase Disk Space : The steps to determine how much space is available on a disk are as follows:

* Double click the My Computer icon and the click the disk you want to check
* Click Properties on the File menu.

A pie chart shows how much free and used space is on the disk.

The steps to create more disk space by using Drive Space to compress both hard and floppy disks are as follows:

i) From the cascading menu of System Tools, select Drive Space option

ii) In the Drives On This Computer list, click the drive you want to compress

iii) Click Compress on the Drive menu.

iv) Click Start

v) If you have not backed up your files, click Back Up Files, and then follow the instructions on your screen. When you are done, proceed to step (vii).

vi) Click Compress Now

vii) If Windows prompts you to restart your computer, click Yes

viii) If you want to free up more disk space after your computer restarts, start the Disk Space Troubleshooter again.

4.2.6.3. Check for Disk Errors : To check your disk's surface, files and folders for errors, follow the following steps :

* From the Cascading menu of System Tools, select of option Scan Disk
* Click the drive you want to check
* Click Through. If you want to change the settings Scan Disk uses when checking the disk's surface, click the Options. If you want to change the settings Scan Disk uses when checking files and folders, click Advanced.
* Click Start button.

4.2.6.4 Format Disks

Formatting a disk means establishing the tracks and sectors on the disk where files will be stored. It is remembered that formating a disk removes all information from the disk. You cannot format a disk if there are files open on that disk. If the disk has been compressed, use Drive Space, to format the disk. To format a disk, the following steps are required:

* If the disk you want to format is a floppy disk, insert in into its drive

* Double-click the My Computer icon, and then click the icon for the disk you want to format. Be sure not to double click the disk icon, because you cannot a disk if it is open in My Computer or Windows Explorer.
* Click the Format on the File menu.
* Click OK button.

The various options for formatting disks are in the following table.

Option	Function
Capacity	It could be low-density or high-density
Format Type : Quick (erase)	Formats the disk without checking for errors
Format Type : Full	Checks for the disk erros and then reformats it
Format Type : Copy System Files Only	Copies the system files to the formatted disk to make it bootable
Label	Type the label name for the disk
No Label	This option is selected if you do not require the label name for the disk after formatting it
Display Summary when Finshed	To get information about the bad sectors (if any) after the disk formatted
Copy System Files	Copies the system files during the formatting process.

4.2.6.5. Set-up user profiles : If more than one person uses the same computer, each one can cutomize the settings according to his/her needs. In this case, each user has to create a profile so that when he/she logs on, their personal windows settings are used. The steps to set-up user profiles are as follows:-

* Click the Password icon from the Control Panel

  The Password properties dialog box is displayed.
* Choose the 2nd option under User Profiles. When it is selected, the options under the head User Profile Settings are highlighted.

- Choose the options as you reqire
- Click on the OK button.

  This enables multiple users to personalize settings.

The steps to log off your computer so someone else can use it are

- Click the Start button, and then click Shut Down
- Then click Close All Programs And Log On As A Different User.

4.2.7 Files and Folders

The File system in Windows 95 is based on folders. Here folders hold files and other folders. Windows Explorer is a file management utility that provides a method of accessing the file system in Windows 95. By Explorer you get a view of your entire file system. To open Explorer, the following steps are follows:

- Select *Programs* from the *Start* menu or right click on the *Start* menu
- Select *Windows Explorer* from the cascading menu of *Programs*

The Exploring window is displayed.

In this Exploring window, there are two parts : left and right. On the left part, you can view the desktop, drives, and folders. When you click a drive and folder, its contents are displayed on the right part. From this window you can copy, move files from one folder to another very easily.

4.2.7.1 Files

Files contain information like text, numeric and graphics. Initially these are created in the memory of the system, but then these are saved to a disk storage device. There are different types of files that are categorized by the type of information they hold and these are discussed below.

File	Type of Information	Example
Program Files	These contain computer readable code written by programmers	Calc. exe, Clock exe
Support Files	Some programs store information in these files, but they cannot execute or started. These files have extension name as OLV, SYS, DRV and DLL	Config. sys,

Text Files	These contain alphanumeric characters that follow the ASCII format	Sample txt., Autoexec.bat
Graphics Files	These contain visual or graphic information	Rivets. bmp
Multimedia Files	Hold sound and video information in digital form	Passport.mid
Font Files	These contain information about various fonts	Coure.foc
Other Data Files	These contain numbers, names, addresses, and other information created by database and spreedsheet programs	

There are different operations on Files, which are discussed below:

Naming Conventios of a File in Windows 95 : For naming a file there are certain rules in Windows 95. These are stated as follows:

- Windows 95 supports 256-character filename in both upper and lower case.
- Files in the same folder cannot have the same name
- You can use multiple period-separated extensions, like Budget. Sales 2006. This is also a strategy for long filenames.
- Names can include spaces but not special characters such as ?*'|<>

Searching a File : The steps for searching a file are as follows:

- Click the Start hutton and the point to Find
- Click Files or Folders
- Type all or part of the file's name in the Named box
- If you do not know the name of a file or want to refine the search, click the Date Modified or Advanced tabs

- If you type upper and lower case letters in the Containing TExt box on the Advanced tab, and you want the search to be case-sensitive, click the Options menu and make sure the Case Sensitive box is checked.
- If you want to specify where Windows should begin its search, click Browse
- Click Find Now
- If you want to save only the search criteria, make sure the Save Results box is clear
- Click the File menu, and click Save Search. An icon representing the search results or search criteria appears on your desktop.

Creating a File : You can use Notepad to create or edit text files that do not require formatting and are smaller than 64 K. Notepad opens and saves text in ASCII (text only) format only. To create or edit files that requires formatting or is larger that 64K, uses WordPad. It is text editor for short documents.

Copy a File : The files can be copied from one destination to another. The steps to copy a file are as follows:

- In My Computer or Windows Explorer, click the file you want to copy. Hold down the CTRL key to select more than one file to copy and click the items you want
- Click the Copy option on the Edit menu
- Open the folder or disk where you want to put the copy
- Click the Paste option from the Edit menu.

The steps to copy a file to a floppy disk are given below:

- Insert the floppy disk in the floppy disk drive
- In My Computer or Windows Explorer, click the file you want to copy
- On the File menu, point to Send To
- Click the drive you want to copy the file.

Move a File : The steps to move a file are given below:

- In My Computer or Windows Explorer, click the file or folder you want to move. To select more than one file to move, hold down the CTRL key, and click the items you want
- On the Edit menu, click Cut

* Open the folder where you want to put file
* Click the Paste option on the Edit menu.

OR

* In drag and drop technique, you use the mouse to click a file or folder and drag it to new location. This technique is used when both the source file and the destination folder, drive or other object to move the source to are visible.

Delete a File : The steps to delete a file or folder are given below:

* In My Computer or Windows Explorer, locate the file you want to delete
* Click the file
* Click the Delete option on the File menu.

Retrieve a File : If you want to retrieve a file you have deleted, look in the Recycle Bin. Your deleted file remains in the Recycle Bin until you emptry it. You can also drag file or folder icons Recycle Bin icon. If you press SHIFT while dragging, the item will be deleted from your computer without being stored in the Recycle Bin.

Finding information on a file : Sometimes you may need to get the following information about certain files, which are itemized below:

* What types of file is this?
* What is its size?
* When was it created or last modified?
* What are the attributes of the file?

To get all of this information, do the following :

* Right-click the file on which you want the information
* Select Properties from the short-cut menu that is displayed

A dialog box is displayed. It gives the information on the selected files.

4.2.7.2. Folders

You now how to create, to delete, to controlling access to a folder.

Creating a Folder : The steps to create a new folder are given below:

- In My Computer or Windows Explorer, open the folder in which you want to create a new folder
- On the File menu, point to the New option, and then click the Folder option

  The new folder appears with a temporary name.
- Type a name for the new folder
- Press Enter

Renaming a Folder : To change the name of folder, follow the following steps :

- In My Computer or Windows Explorer, click the folder you want to rename. You do not need to open it.
- Click the Rename option on the File menu
- type the new name
- Press ENTER

Viewing a Folder : To see the hierarchy of folders on a disk drive, the steps are :

- Click the Start button
- Point to Programs, and then Click Windows Explorer
- Click a folder on the left side of the window to display its contents on the right
- Click the plus signs (+) to display more folders

Note : To change the size of either side of window, drag the bar that separates the two sides. To quickly open a folder and display its sub-folder, double-click the folder on the left side of the window.

Deleting a Folder : Deleting a folder removes all files and subfolders within it, follow the following steps:

- In My Computer or Windows Explorer, locate the folder you want to delete
- Click the folder
- Click the Delete option on the File menu.

OR

- Right-click the folder and select Delete from the menu

OR

* Right-click and drag the folder to the Recycle Bin.

The deleted files can be retrieved from the Recycle Bin. But if these are removed from recycle bin also, then it is not possible to recover the deleted folders.

Controlling access to a Folder : You can control access to your folders by using passwords or by listing names of people you want to have access these resources. To control access to a folder, the steps are as follows:

* In My Computer, click the shared folder you want to limit access ot
* On the File menu, click Properties
* Click sharing
* If you are using user-level access control, click Add to specify the people you want to be able to use your resources
* If you are using share level access control, type the password you want to use for the folder.

4.2.7.3. Recycle Bin

The Windows 95 Recycle Bin protects you from accidentally deleting files. This is located on the desktop. You can use it retrieve files you deleted by mistake, or empty it to create more disk space. The Recycle Bin holds deleted files in a queue, with the most recently deleted files on the top. The oldest file is permanently deleted when the queue becomes full, as Recycle Bin has a limited amount of space. The deleted files can be recovered by dragging them out of the Recycle Bin to an appropriate drive and folder. The Recycle Bin folder has a menu with many familiar options, for example, the View option is to change the way files are listed or rearrange the order of files. To clear out the Bin, choose Empty Recycle Bin from the File menu. Selecting them and choosing Delete on the File menu can also delete individual files. This will permanently delete the files without any chance of recovering them.

4.2.8 Program and Accessories

Windows 95 comes with a set of programs called Accessories. These accessories are used to write, paint, calculate, and perform a variety of other tasks. To run the accessory programs, follow the following instructions:

* Click the Start button

* Click on Programs and then click on Accessories option
* Click on the program you want to run from the cascading menu.

The steps to run other programs that have been created by the users are as follows:

* Click on the Run option from the Start menu

  Run dialog box is opened.
* Enter the path name of the program you want to run, in the Open text box.
* If you do not remember the name of the file, click on the Browse button
* In the Browse window, look in text bos lists the available drives, folders, and files. Click on the down-arrow.
* To see inside a folder, click it. The box below shows the folders and files in the selected location
* Double-click a folder or file to open it
* Once the file is located, click on Open and then click on OK from the Run window.

Calendar : Windows Calendar is used to view the current month's calendar that shows the day and date, and also the current day showing the timings to keep track of your appointments. To display either of the mentioned views, do the followings :

* Click on the Viw option
* Select whichever you require.

Calculator : You can use Calculator in standard view to perform simple calculations, or in scientific view to perform advanced scientific and statistical calculations. To start the Calculator, do the following :

* click the Start button
* Point to Programs from the Start menu, then pointing to Accessories
* Click the Calculator option.

Character Map : It works only with Windows based programs. This is used to insert a special character into a document. The steps to insert a special character in the document :

* Select the Character Map from the cascading menu of Accessories
* Click the Font box, and then click a font

* To magnify a character in Character Map, click it and hold down the mouse button. Double-click each character you want. The character(s) will appear in the Characters To Copy box
* Click The Copy option
* In your document, click where you want the character(s) to appear, click the Edit menu, and then click Paste option.
* Select the character(s), and then change them to the same font you used in the **Character Map**

Paint : It is a bitmap painting program with a full set of painting tools and a wide range of colors. You can use it to create, edit and view pictures. The picture created by Paint can be pasted into another document you have created or use it as your desktop background. To work with Paint, do the following steps :

* Select the Paint option from the cascading menu of Accessories

The Paint window is opened which,consists of a workspace, or canvas, where you paint pictures. To the left of the canvasa is the Toolbox, which contains a set of painting tools. The color palette is at the bottom of the canvas. The selection box is on the left side, where you select the width of lines or pen tips to use for the Bursh, Line, Eraser, and other tools.

* To paint, select a tool, a color, and a line, and the start painting on the canvas.

WordPad : It is a text editor for short documents. To start WordPad, do the following:

* Click on the Start menu
* Point to Programs, and the click on the Accessories option
* Choose the WordPad from the cascading menu

When WordPad is running, you can do the following:

* Specify the initial page layout for a document, such as paper size and margin settings
* Type, edit, and delete text, as well as copy or move text from one place to another.
* Change the font, style, and size of characters, and change the alignment and indents of paragraphs.
* Search for previously typed text or replace text with new text.
* Create compound documents that contain pictures, spreadsheet data, charts, sound, and video created in other applications.

Notepad : You can use Notepad to create or edit text files that do not require formatting and are smaller than 64K. Notepad opens and saves text in ASCII (text only) format only. To create or edit files those require formatting or are large than 64K, use WordPad. To start, follow the following:

* Click Notepad from the cascading menu of Accessories.

5.0 Unit Summary

From this module, you can learn all basic operations such as copy, move a file, to create a folder, to delete a folder, to rename a file and folder, to get the file information, to increase the memory size, to back up utility, to check all drives, to access accessories, etc. in Windows 95.

6.0 Self Assessment Questions

1. What are the parts of Windows?
2. How to set up the screen saver?
3. Name the various disk drive utilities.
4. How can you add Windows 95 components or accessories that were not installed before?
5. What is the function of Windows Explorer?
6. What is a Recycle Bin?
7. What are the Accessories of Windows?
8. Change the mouse double click speed.
9. Show how to retrieve the file from dustbin.
10. Add one program in Start menu.

7.0 Suggested further Readings

1. Microsoft Windows 95 manual
2. IGNOU Books

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VIDYASAGAR UNIVERSITY

DIRECTORATE OF DISTANCE EDUCATION
MIDNAPORE - 721 102

M.Sc. in Applied Mathematics with Oceanology and Computer Programming

Part-I

Group-B Paper-V

Module No. - 56

MICROSOFT WORD

CONTENTS

1.0 Introduction

MS-WORD 95 is a word processor, which is a software package that helps you to create and edit a document much faster than the usual manual ways. The following functions can be accomplished by a word processor.

- Creating and typing a document through the keyboard
- Saving the document
- Correcting, deleting and inserting characters, words, lines and images
- Opening an existing document
- Moving or coping paragraphs or images from one place to another
- Changing the text font and style
- Checking for spelling errors
- Printing the document in various formats

2.0 Objectives

The different techniques are discussed in this unit, by which a document can be prepared completely and easily.

3.0 Key Words and Study guides

Editing, Saving, Opening, Coping, Pasting, Moving, Alignment, Front, Printing, Spell Checking

4.0 Main Discussion

4.1 Starting MS-WORD

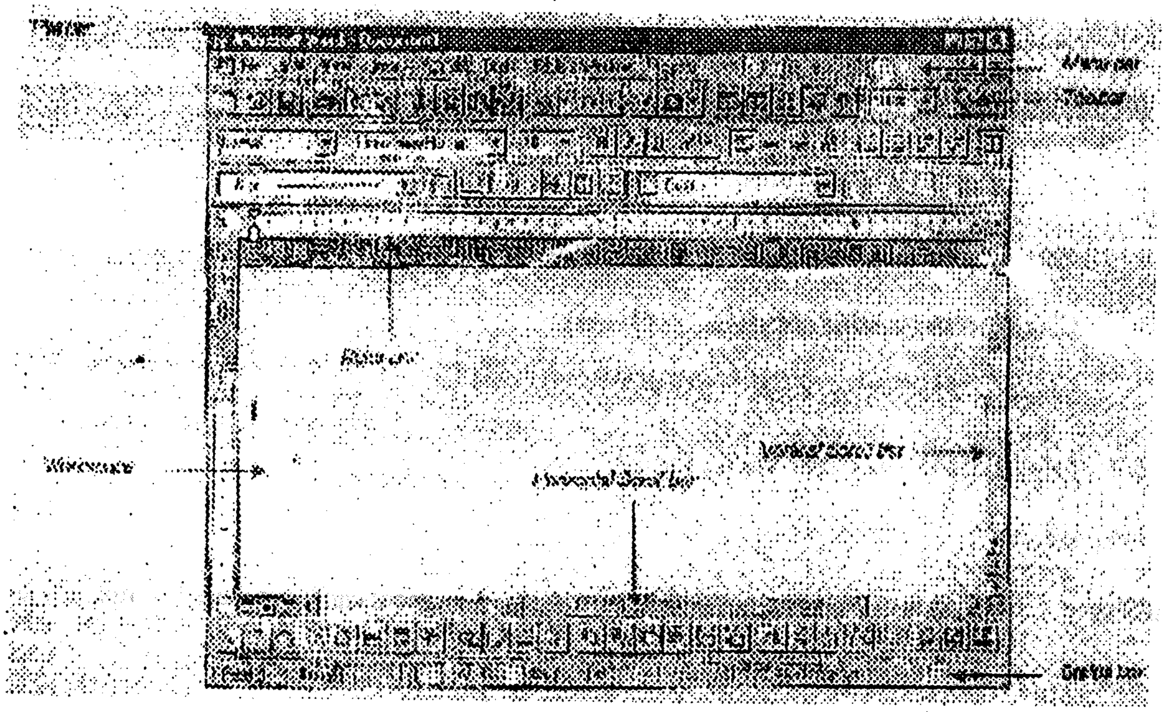
To invoke MS-WORD first you see the icon Microsoft Word on the desktop. If there is the icon then double click on the icon. This displays a document window. Otherwise follow the following steps to invoke MS-WORD.

- Click on the start button in the taskbar.
- Select the Programs option from the start menu.
- Clic on the Microsoft Word option or Click MS Office group and then Click on the Microsoft Word.

This invokes MS-WORD and simultaneously opens a word document window.

4.1.1 The First MS-Word Screen

When the word is started, a new and blank document appears on the word screen. The various parts on this screen are shown below:



The Word document window has the following basic components:

- Title bar : It displays the name of the program, the name of the currently active word document, the control menu icon, the maximum button, the minimize button and the Restore and close buttons.
- Menu bar : It is positioned below the Title bar. It contains options like File, Edit, View, Insert, Format, Tools, Window and Help. each of these menu bar items has a drop-down menu.
- Toolbar: It contains the buttons that provide quick access to commonly used commands. Clicking a button has the same effect as selecting the command from the menus, but can be donw in a single action.
- Ruler bar : It allows you to accurately set the layout of the document. It allows you to set tabs, indents, and change page margins.

- Status bar : It displays information about the active document or the task on which the user is currently working. This includes the page number, the column and the line number on which the cursor is positioned and so on. The status bar is positioned at the bottom of the window.
- Scroll bar: It helps to scroll the contents of a document. The Word document consists of two scroll bars:
  - (i) Vertical scroll bar : THe vertical scroll bar is used to move a document vertically. It has four buttons, those with single arrows allow the user to scroll up or down in the document one line at a time while those with double arrows allows the user to scroll by one page.
  - (ii) Horizontal scroll bar : It is used to move the document horizontally.
- Workspace: It is the area in the document window where the user can type the text of the document.

4.1.2 To Create a Document

To create a new document at any time, follow the following steps:

- Select the New option from the File menu or click the new button on the standard toolbar. A new dialog box will be displayed.
- Select the Blank document Icon.
- Click on the document radio button in the Create New box.
- Click on the OK button.

  A blank new document is displayed.

When you open a document it is placed on the top of any existing once and becomes the active document.

4.1.3 To Open a Document

To quickly open one of the last documents you worked on, choose it from the File Menu. Generally to open the existing document the following steps are required:

Select the Open option from the File menu.

An Open dialog box will be displayed.

- Select the appropriate drive and folder in which place the file is saved.
- Type the name of the file to be opened in the File name box or select the file from the list.
- Click on the Open button.

4.1.4 To Edit Text

When you open a new document in Word, the insertion point is at the top of the document, ready for you to begin typing. Typing text in MS-WORD is very easy and straight forward task, because of the useful word processing features supported by it. When you create a new document in Word you can just start typing. Press the Enter key only when you have finished a paragraph. MS-Word automatically moves the text to the next line when it reaches the right edge of the screen. This feature is known as **word-wrap.**

When you enter your text, you are quite likely to make mistakes. Corrections of these mistakes aer called editing. Editing text includes selecting, deselecting, deleting, inserting, replacing text. Before editing can begin, the insertion point (cursor) needs to be positioned at the location where the changes have to be made. To move the cursor the following key and key combinations are required:

Name of Key(s)	Moves the Cursor to
Up arrow, Down arrow, Left arrow, Right arrow	One character up, down, left and right
Ctrl + right arrow	Next word
Ctrl + left arrow	Previous word
Home	Beginning of the line
End	End of the line
Ctrl + Home	Beginning of the document
Ctrl + End	End of the document

Selecting text: In order to perform any editing operation on the text in a document, the text needs to be selected. The steps to select a text are as follows:

- Position the insertion point at the beginning of the text to be selected.

- Hold down the left mouse button and drag the mouse in any direction across the text to be selected.

The selected text will be highlighted.

Deleting text: Text to be deleted by using either the "Delete" or "Backspace" keys. Delete removes single character to the right of the cursor, while backspace removes single character to the left. Block deletions make it easy to delete sentences, paragraphs and sections of documents. Once you have selected the text you wish to remove, pressing the "Delete" or "Backspace" key removes it from the document.

Moving text: To move the text from one location to another, do the followings:

- Select the text to be moved.
- Select the cut option from the Edit menu.
- Position the insertion point at the location where the text should appear.
- Select the Paste option from the Edit menu or click on the Paste button on the standard toolbar.

OR

The drag and drop feature can be used to move the text as follows:

- Select the text to be moved.
- Point to the selected text and holding down the mouse button, drag the selection to the desired location.
- Release the mouse button to drop the text.

Inserting text: MS-WORD is normally in the insert mode. It offers several ways of inserting text into an existing document. The simplest among all, is to move the cursor to the required position and start typing. The existing text will be pushed and adjusted accordingly.

Replacing text: Word can combine the steps of deleting unwanted text, positioning the insertion point, and inserting replacement text. To replace text, do the following:

- Select the text to be removed
- Start typing the new text
- The selected text will be removed and the new text accommodated.

Undoing Edits: The Undo command in the Edit menu is used to reverse actions. For example, while deleting text the user presses the "Delete" key. After deleting immediately if you wish not to be deleted then click Undo option from the Edit menu.

Coping and Moving: Coping means to make a copy of the selected text or graphics and insert it in another location, leaving the original unchanged. The steps to copy text to another part of the same document or in a different document are as follows:

- Select the text to be copied
- Right click on the highlighted text.

  A shortcut menu appears near the highlighted text.
- Select the copy option from the shortcut menu.
- Position the cursor at the location where the text is to be pasted.
- Select the Paste option from the shortcut menu.

If a portion of the text is to be moved to a different place, select the text to be moved and then use the cut option from the Edit menu. The selected text will be copied and deleted. Now the Paste option is enabled in the Edit menu. To paste the text, place the cursor at the starting point of insertion and choose Edit menu and then Paste option. The text will be pasted there.

Changing the Case of text: The "Change Case" command in the "Format" menu allow you to change the Case of the characters of the characters in text without having to retype them. The steps to convert the case of the text are:

- Select the text to be changed
- Select the Change Case option from the Format menu

  A Change Case dialog box is displayed.
- Select the desired option
- Click on the OK button.

There is different option on the Change Case dialog box, which is discussed below:

Option	Function
Sentence case	Capitalizes only the first letter in the selected sentences
Lower case	Converts all selected text to lower case letters
Upper case	Converts all selected text to upper case letters
Title case	Capitalizes the first letter of each word of the selected text
Toggle case	Changes upper case to lower case and lower case to upper case in all selected text

Creating Bulleted and Numbered lists: You can easily create a bulleted or numbered list by selecting a list and then clicking the "Bullets and Numbering" button from the "Format" menu. Alternatively, you can click either of these buttons before you type a list. When you have finished typing the list, just press ENTER.

Formatting Text

The formatting features in Word sets each document apart from others. Formatting a document includes assigning fonts and font sizes, aligning text, dividing text into columns, adjusting the line and paragraphs, and setting margins. All these are discussed below:

Changing Font Styles: Font refers to the manner or style in which text is displayed in the document. Different fonts contain different collection of characters and symbols. The procedures are given below:

- Select the text whose font has to be changed.
- Click on the down-arrow to the right of the Font list box in the formatting tool bar.

  A list of available fonts is displayed.
- Select a font from the list.

Changing Font Size: Font size is measured in points. Points and picas are used for measuring spacing, line thickness, and so on. There are 12 points to a pica and six picas to an inch; therefore, there are 72 points to an inch. The steps to change the font size are:

- Select the text whose font size has to be changed.
- Click on the down-arrow to the right of the Font Size list box in the formatting tool bar.

  A list of available font sizes is displayed.
- Select a font size from the list.

Alternatively, Font style and Font size both can be changed simultaneously by Font dialog box. The required steps are given below:

- Select the Font option from the Format menu

  The Font dialog box is displayed.
- Make the appropriate selections in the Font dialog box.

In the Effects area, a black check mark on an option indicates that the selected text exhibits that particular attribute. A completely clear box for an option indicates that the selected text does not have the specific attribute.

Color Choice: To change the color of the text, do the following

- Choose the text of interest
- Choose desired color from the Color option box in the Font tab of the Font dialog box of Format menu.
- Click on the OK button.

4.1.5 To Save a Document

When a document is typed, it is stored in the main memory of the machine. In order to preserve the document for future use, it needs to be saved on the auxiliary memory i.e. on the disk. The steps to save a document are as follows:

- Select the Save option from the File menu.

  The Save As dialog box is displayed.
- Type a file name in the File name box.
- Click on the Save button

By default, the new document is saved as a Word document with an extension .doc.

4.1.6 To View a Document

There is different view level to make a document easier to visualize which is discussed below:

- Normal View: Normal View is the default view in Word. It is used for most of the typing and editing. The area outside the text body-the area containing headers, footers, footnotes, page numbers and margin spacing-does not appear in the normal view. To display the normal view, select the Normal option from the View menu.
- Outline View: Outline view enables you to view the entire contents of the documents, only chapter headings or only section headings. Using the outline view, you can quickly move through a document and rearrange several lines of text by moving the headings. To display the outline view, select the Outline option from the View menu.
- Page Layout View: It shows each page of the document as it will appear when printed. The users can scroll outside the text body to see such items as headers, footers, page numbers and footnotes. To display the page layout view, select the Page Layout option from the View menu.
- Master Document View: It allows you to group multiple word documents into one large document. To display the master document view, select the Master Document option from the View menu.
- Full Screen View: It is used when you want to maximize the typing area. The title bar, menu bar, toolbars, scroll bars and status bar are removed from the screen in full screen view. To display the full screen view, select the Full Screen option from the View menu.

4.2 To Format a Document

4.2.1 Line Spacing

The steps for line spacing are as follows:

- Select the paragraph(s) to space
- Select the Paragraph option from the Format menu.
  The Paragraph dialog box is displayed.
- Select the Indents and Spacing tab.
- Select an option from the Line Spacing drop-down list.
- Select a value from the At list box.
- Click on the OK button to close the dialog box.

Look at the Preview area to check how the formatted text would appear.

4.2.2 Paragraph Spacing

Word enables each paragraph to give unique before and after spacing if you wish. The spacing settings can be in points (pt), inches (in), centimeters (cm) or lines (li). One advantage to adding space this way is that the spacing before and after paragraphs does not change the point size of your text. Another advantage is that you can use different spacing combinations for different purposes. You can define the amount of white space that should be placed before and after paragraphs by using the Paragraph dialog box. The steps to add a single line of white space before or after a paragraph are as folows:

- Select the paragraph(s)
- Open the Paragraph dialog box clicking the Paragraph from the Format menu
- Select the Indents and Spacing tab.
- Enter the specification in the Before or After Spacing spin box
- Click on the OK button.

4.2.3 Setting Tab Stops

Working with tabs is a two-part process. The first step is to set the tab stops. Word offers five types of tab stops - Left, Center, Right, Decimal and Bar tabs. The default tab stop is left. The second step in using tabs is to press the Tab key as the document is typed. By default the tab stops are set at 0,5 inch internals from the left margin. The insertion point can be moved to the next tab stop in the current paragraph by pressing the TAB key. To set tab stops do the following:

- Select the paragraph in which you want to set or change tab stops
- Select the Tabs option from the Format menu
- Using the decimal numbers, type the position of the tab stop in the Tab Stop Position box
- Select the tab leader style you want: 1 for no leader, 2 for a dotted leader, 3 for a dashed leader, or 4 for an underfined leader from the Leader group
- Click on the Set button to set the tab stop
- Click on the OK button.

To move a tab stop, point to the tab marker and drag it to a new position. To clear a tab stop, either drag the tab marker off the ruler or, use the Clear button on the Tab dialog box to clear the tabs.

4.2.4 Indenting Text

Indenting is making your text look more eye-catching. Indents are added to margins, thereby increasing the white space and decreasing the text area for specific paragraphs. Indent can be performed in two ways:

- By Ruler
- By Paragraph dialog box

By Ruler: The steps to set the left and right indents with the ruler are as follows:

- Select the paragraph(s) for which you want to set tabs
- Point to and then drag the appropriate triangular indent markers to the desired locations. When you release the mouse button, the text will be indented according to the specification.

By Paragraph dialog box: Using it, you can specify the left, right, first line and hanging indents. The steps to set indents are as follows:

- Select paragraph(s) to be indented
- Select the paragraph option of the Format menu

  The Paragraph dialog box is displayed.
- Enter the specifications in the Left or Right Indentation spin box

  You can also select First Line or Hanging indent from the Special drop-down list box.
- Select a value in the By spin box.
- Clic on the OK button.

4.2.5 Borders and Shading

You can add borders to any side of a paragraph and you can add background shading. You can also add borders and shading to ordinary text and to the paragraphs in table cells and frames. Clicking Borders and Shading from Format menu can do this.

4.2.6 Aligning Text

Alignment is a way of organizing the text. It refers to the position of the text relative to the margins. Word enables the user to left-align, right-align, center-align and justify the text in the document in order the text in the document in order to enhance it. Using the four formating toolbar buttons can do this. By default, Word uses left alignment.

Left aligned: Text is said to be left aligned if it is aligned with th left margin of the page. This is the default mode of alignment.

Right alignment: Text is said to be right aligned if it is aligned with the right margin of the page. To right align a paragraph, position the cursor on any line within the paragraph and click on the Align Right button on the Formatting toolbar.

Centered: The center option is normally used to center the heading and text. To center a line of text, position the cursor on the line and click on the Center button on the Formatting toolbar.

Justified: This feature aligns a paragraph with both the left and the right margins. Inter word spacing is adjusted such that each line of text beings at the left margin and ends at the right margin. To justify a line of text, position the cursor on that line and click on the Justify button on the formatting toolbar.

4.2.7 Section Break

Section breaks divide a document into sections. The breaks appears as double-dotted lines containing the words End of Section in Normal view. The following steps to insert a section breaks are:

- Position the insertion point at the location where you want to insert a break
- Select the Break option from the Insert menu

  The Break dialog box is appeared.
- Select one of the following options in the Section Breaks are of the Break dialog box.
  - Next page radio button to insert a section break and begin the new section at the top of the next page
  - Continuous radio button to insert a section break and begin the new section on the same page, below the previous section

- Even page radio button to insert a section break and begin a new section on an even-numbered page
- Odd page button to insert a section break and begin a new section on an odd-numbered page.

• Click on the OK button to close the Break dialog box.

To delete a section break, select the section break and press the del key.

4.2.8 Paging

Word offers a variety of tools to help you automatically number pages. You can choose from many page numbering format and style choices and position page numbers nearly anywhere that pleases you. If you plan to break a document into multiple sections, you may want to insert page numbers before you split the document into sections. Otherwise, you will have to repeat the page numbering for each section of your document.

Page Breaks: Word uses "Soft" and "Hard" page breaks to indicate when one page ends and another begins. There are two categories of page breaks:

- Soft Page Break: Word automatically inserts it, when text reaches the bottom of the page and is represented by a dotted line across the document. The text beneath it will appear on the second page if we print this document. The position of Soft Page Breaks in a document automatically changes when text is added or deleted to reflect the new page boundaries.
- Hard Page Breaks: Your insert Hard Page Break when you want to end one page and begin another. This is done via the "Break" command in the Insert menu or by pressing "Ctrl+Enter". Hard Page Break appears with the words "Page Break" on the line. They remain at the exact same spot in the document regardless of any text changes you make. To remove a hard page break you position the cursor just after it and press the backspace key, or position the cursor at the start of the page berak and press the "Delete" key.

The page number are placed in a frame in the header or footer area. There are two methods, which aer as follows:

- Using The Header and Footer Toolbar method: The steps to add page numbers by this technique are as follows:
  - Open the header and Footer area by double-clicking on the header or footer in the document.

- Click on the Insert Page Number button in the Heder and Footer toolbar.

  A number is inserted in the header or footer area at the insertion point.

- Using the Page Numbers option: This option provides a variety of numbering options including: Page numbering formats and styles and Suppressing first page numbers. The steps to insert page numbers are as follows:

  - Select the page Numbers from the Insert menu.

    The Page Numbers dialog box will be displayed.

  - Specify the position and alignment for the page number using the Position and alignment drop-down boxes, respectively.
  - Put a check mark in the Show Number on the First Page Check box.
  - Click on the Format button in the Page Numbers dialog box.

    The page Number Format dialog box will be displayed.

  - Make the appropriate selections in the Page Number Format dialog box.
  - Click on the OK button to close the Page Number Format dialog box.
  - Click on the OK button to close the Page Number dialog box.

4.2.9 Additing Headers and Footers

Headers and footers contain text that needs to be displayed on each page of a document. Headers get displayed at the top while footers at the bottom of a document. You can use identical headers and footers on all pages in your document, or you can specify different contents for each section of the documents. Odd and even pages can have different designs if you wish. To enter a header that repeats on all pages in your document, do the following:

- First choose View from the menu and then Header and Footer

  The Header and Footer toolbar is displayed.

- Create and edit header text. You can also paste graphics.
- Click on the Close button or double-click in the main document.

The steps to add footer are as follows:

- Select the Header and Footer option from the View menu.

- Type or edit the footer text.
- Click on the close button.

4.2.10 Paper Size and Page Orientation

The paper size tab in the Page Setup dialog box chosen from the Page Setup in the File menu can be used for selecting the paper size. The options that are available depend on the capabilities of the printer you have selected. To use a custom paper size, type the dimensions of the paper you want to use and also be sure that the printer is capable of feeding the custom paper size through its printing mechanism.

Page orientation can be vertical (Portrait) or horizontal (Landscape). To select the paper size and page orientation follows the following steps:

- Select the text you want to have a different paper size or page orientation.
- From the File menu, choose Page Setup and then select the Paper size tab.
- Select the paper size on which you want to print and the page orientation.
- In the Apply To box, select how much of the document you want to print on the selected paper size or in the selected orientation.
- Then click the OK button.

4.2.11 Paper margin

Margin determine the distance between the text and the page of the paper. In Word, text and graphics are normally printed inside the margins while headers; footers and page numbers are printed in the margins. The most straight forward method is as follows:

- Place the insertion point on the page where you want margin settings to be changed (unless you plan to see the whole document choice).
- Click the Page Setup from the File menu.

  The Page Setup dialog box is displayed.
- If need be, change paper size and orientation by using the Paper Size tab.
- Switch to the margins tab if it is not already displayed.

- Current settings are shown in the various margin dimension boxes.
- Type the dimensions you desire, or click the little triangles to increase and decrease settings. The Preview will change as you work.
- When satisfied, click OK.

Mirror (Facing) Margins: Select the Mirror Margins feature in the Margins tab of the Page Set up dialog box when you want different left margins widths and your final output will be two-sided. Word makes inside margins of odd and even numbered pages the same size; and does the same with the outside margins of odd and even page. When adjusting margins in Print Preview, if you have chosen the Mirror Odd/Even feature.

4.2.12 Bookmarks

Bookmarks are used to identify the beginning of a chapter, tables or the place in the document where you left off. You can mark a section, a chapter, a range of characters, graphics or any other Word element. Bookmarks can be used to jump to specify points in a document without having to scroll or search through the pages in the document. To create a bookmark, the following steps are

- Position the insertion point at the location where you want to create a bookmark.
- Select the Bookmark option from the Insert menu.

  The Bookmark dialog box is displayed.
- Type a name for the bookmark in the Bookmark Name text box
- Click on the Add button

Bookmark names can obtain up to 40 characters. A name must begin with a letter and can include numbers, letters and underlines but not spaces, punctuation marks or other characters. The steps to go to a predefined bookmark are as follows:

- Select the Go To option from the Edit menu.

  The Go To dialog box is appeared.
- Choose the bookmark option from the Go To what scroll list.
- Choose the name of the bookmark from the Enter Bookmark Name drop-down list

- Click the Go To button.

To delete the bookmarks the following steps are requierd:

- Choose the Bookmark option from the Insert menu.
- Select the name of the bookmark to be deleted.
- Click the Delete button.
- Click the Close button.

4.2.13 Table

Word's Table feature enables you to arrange columns of numbers and text in a document without using tabs. It consists of horizontal rows and vertical columns. The intersection of a column and row is a rectangular or square box called a cell. You do typing in the cells. Cells can contain text, numbers, or graphics. A number of table specific features let you control the size, shape, and appearance of cells. Border and shading features are available. It is also easy to insert the delete rows and columns. The text wraps to the next line according to the width of the cell. If the width of the cell is adjusted, the text also adjusts to the new width.

Creating a Table: To insert a table in a document the following steps are required:

- Place the insertion point where you want to insert a table.
- Choose the Insert option from the Table menu

  The Insert Table dialog box is appeared.
- Mention the number of columns and rows for the table in the Number of Columns and Number of Rows text boxes respectively.

  You can use the Auto Format button to apply predefined formats to the table when Word creates it.
- Click the OK button.

Entering and editing text in a table: You navitage, enter and edit table text just as you do any other Word text. The mouse or arrow keys are used to position the insertion point. The cells are thought of as miniature pages and the cell borders as margins. Type the text normally within these cells and Word will automatically wrap text within the cell as you reach the right edge Rows will automatically grow taller as necessary to accommodate your typing. Remember the followings:

- To move from cell to cell within a table, either use the **Tab** key to go forward and **Shift + Tab** to go backward.
- Pressing **Tab** in the right most column will move down the insertion point to the beginning of the next row and pressing **Shift + Tab** past the left most column will move the insertion point to the end of the previous row.
- A cell can contain more than one paragraph. Paragraph creation is done in the usual way.

Converting Tables to Text: To convert an existing table to text, do the followings:

- Select the table cells you wish to convert, or Alt+double click to select the whole table
- Choose the Convert option from the Table menu
- Choose the Table To Text option

  The Convert Table To Text dialog box is displayed.
- Choose one of the options: paragraph marks, tab-delimited text, comma-delimited text, and others. The paragraph option will convert each old table cell into at least one paragraph. If the table's cell contains multiple paragraphs, the paragraph marks are retained during conversion. If you pick the comma or tab options, Word will convert each row of your table into the paragraph. Cells from the tables will be separated within the paragraphs by tabs or commas.
- Click on the OK button.

Converting Text to Table: To convert text to a table, do the followings:

- Highlight the text you want to turn into a table
- Choose the Convert option from the Table menu.
- Choose the Convert Text To Table option.

  The Convert Text To Table dialog box is displayed.
- Click on the appropriate option button.

The Convert Text To Table contains different options. Among these some are explained below:

Tab: Lines of text separated by paragraph marks or the breaks will become rows in your table. Tab-separated strings of text within those lines will become cell entries in the row. Word will automatically create the necessary number of columns based on the maximum number of tabs in a line.

Comma: Lines of text sepated by paragraph marks or line breaks will become rows in your table. Comma-separated strings of text within those lines will become cell entries in the row. Word will automatically create the necessary number of columns based on the maximum number of commas in a line. Beware of commas that might create unintemational cells.

Paragraph: Word will propose a single column and create as many rows as you have paragraphs. Changing the number of columns will distribute paragraphs among the columns from left to right. In a two-column lay-out, the first paragraph would end up in the top-left cell of the new table, the second paragraph in the top-right cell, the third in the left cell of row two, and so on.

Modifying a Table: Modifying a table involves selecting, inserting, deleting, copying and moving rows, columns and cells. It also describes how to change the spacing and column width and how to split a table. The steps to insert a row in the table are as follows:

- Select the row where you want a new row to be inserted
- Select the Insert Rows option from the Table menu
- Click on the OK button.

When a row is inserted, Word shifts the selected row down to make room for the inserted row. The steps to insert a column in the table are given eblow:

- Select the column where you want a new column to be inserted
- Select the Insert Columns option from the Table menu
- Click on the OK button.

When a column is inserted, Word shifts the selected column to the right to make room for the inserted column. The steps to delete a row or a column are as follows:

- Select the row or column to be deleted
- Select the Delete Rows or Columns option from the Table menu

The Merge Cells feature is used to combine multiple cells. To merge multiple cells into a single row or single column, do the following:

- Select the cells to be merged
- Select the Merge Cells option from the Table menu

To split a cell the following steps are required:

- Select the cell
- Select the Split Cell option from the Table menu

  The Split cell dialog box is displayed.
- Type the number of rows and number of columns in the Number of rows and Number of column Text Box
- Click on the OK button.

Adding Borders and Shading to a table: The steps to add borders to all or selected parts of a table are as follows:

- Select the entire table or the cells that you want to border

  Select the Borders and Shading option from the Format menu

  Select a line style from the Style list

  Select a color for the line from the Color list

  Select one of the border patterns in Setting group

  Click on the OK button

4.2.14 Cross-Referencing

It refers to referencing information in other parts of your document. Word can track the relevant references when things change. For example, the user may type the following text: "For more information see page" and insert a cross-reference to a page number, heading text or heding number. To create a cross-reference, do the following:

- Type the introductory text preceding the cross-reference
- Select the Reference option from the Insert menu and then select the Cross-reference option

  The cross-reference dialog box is displayed.
- Select the information you want to insert in the document from the Insert Reference To list
- Select the specific item you want to from the For Which numbered item list box
- Click on the Insert button to insert the cross-reference.

To update cross-reference, select the entire document and press F9.

4.3 Word Tools

4.3.1. Mail Merge

Mail Merge feature lets you quickly creaet personalized correspondence and other documents by combing information (merging) information from two different files. In a Business sceneario, it is often required to send letters with identical information to a group of people who erside at different locations. The letters may require the address of each receipient to be printed on the top in addition to the standard information. This problem is taken care of by the Mail Merge feature of Word. Mail Merge feature can also be used to prepare other kinds of merged documents such as catalogs, parts list, directory list forms etc. It is used to combine a data source with a main document.

Data Source and Main Document: Data Source is organized collection of information-database-stored as Word table. Word can also use data from other applications such as MS Excel or MS Access. After opening a data source in another application, make sure that the merge fields in your document match those in the data source. All data sources, no matter where they come from, contain records and fields.

Example: An employee data source would contain one record for each employee. This record would contain multiples fields-one for the employee's first name, one for the middle name, one for the last name, and one for each part of the address and so on.

You can either open an existing data source or create a new one in Word.

Main Document contains the text of the body of the letter, fields, and merges instructions. It can be used from the Word or from the other applications. While using the Main document from the other applications open that documents in Word6 and convert its contents to Word for windows. However, field names and formatting from some applications may not translate well into Word for Windows format.

Performing a Mail Merge involves **three basic steps** which are as follows:

- Creating the main document, i.e., to set up the main document, which contains the text, punctuation, and other items that remain same in each version of the form letter.
- Creating the data source, i.e., to set up a data source, which contains the information that varies in each version. This can be done either by opening an existing data source or creating a new one.
- Merging the data source and the main document.

Creating a Main Document: The steps to create a main document are as follows:

- Select the Mail Merge option from the tools menu

  The Mail Merge Helper dialog box is displayed.

- Select the Create button to start creating your main document.

  The Create drop-down list is displayed. In this a list will drop down offering you four choices:

From letters, Mailing Labels, Envelopes and Catalogs.

- Select the type of main document you want. Suppose you select Form letters from the list of displayed options

  The Microsoft Word dialog box is appeared.

- Click on the Active Window button if the active window contain the information for your main document or click on the New Main Document to open a new document as the main document.

Creating Data Source: Next, you need to specify the data source and arrange it in the fields that will be available to your main document for the merge. The steps to specify a data source are as follows:

- Click the Get Data button in the Data Source area of the Mail Merge Helper dialog box.

  A Get Data drop-down list which contains a list of options for data source is displayed.

- If you already have a data source that you want to use, select Open Data source or if you want to create a new one, select the Create Data Source option from the Get Data drop-down list.

  A create Data Source dialog box is appeared. This dialog box contains a list of commonly used field names.

- To create a new field, type the new field name in the Field name text box and then click on the Add Field Name button to add the new field name to the end of the Field Names in Header Row list box.

- To move an existing field name, select it in the Field names in the Header Row list box and use the Move buttons to move its position in the list. The Remove Field Name button can be used to delete a selected field name.

- Click on the OK button.

The Save As dialog box is appeared.

- Type the name for the data source and click on the Save button

The Microsoft Word dialog box gets invoked.

- Select the Edit Data Source button to add records to the data source

The Data Form dialog box is displayed.

- Type the appropriate information in all the text boxes.
- Click on the Add New button of the Data Form dialog box to add another blank record and enter the information for that record.
- Select the View Source button to display the data source document.
- Choose the OK button when you have finished entering all records.

Merging data source and main document:

After you have created data source, you can complete the main document by inserting merge fields into the main document. This can be done by two steps:

- Editing the Main document
  - Select the Mail Merge option from the Tool Menu
  - Click on the Edit button in the Main document area of the Mail Merge Helper dialog box.
  - Select the file name from the Edit menu in the Mail Merge Helper dialog box to active the main document.
  - Type or add any text and graphics you want to include in the main document window.
  - Position insertion point where you want to add a merge field.
  - Click on the Insert Mail Merge field button on the Mail Merge toolbar and select the field to be inserted.
  - Click save button of the standard toolbar to save the main document.

- Merging the Data Source with the Main document
  - Oopen the main document.
  - Click the View merged Data button on the Mail Merge tool bar while the main document is in the active window.
  - Select the Mail Merge option from the tools menu to invoke the Mail Merge Helper dialog box.
  - Click on the Merge button in the Merge the Data with the Document area to begin the merge process.

    The Merge dialog box is appeared.
  - Select a type of merge from the Merge to drop-down list
  - Select the number of records to be merged in the Records to be merged group.
  - Click on the Merge button after you have made the appropriate selections in the Merge dialog box.

You get the complete merged documents.

Creating labels and envelops: Word's Mail Merge Helper can also be used to merge labels and envelops. The procedures for merging labels and envelops are very similar to those for form letters. Try it.

4.3.2 Macros

A **Macro** is a series of Word commands grouped together as a single document to make everyday tasks easier. It can be assigned to a toolbar, a menu, a shortcut key and run it by simply clicking a button, selecting a menu choice or pressing a key combination. Macros are instructions in word's macro language, Word Basic. The uses of Macros are as follows:

- Recording a Macro
- Editing and organizing a macro
- assigning macros to a menu, toolbar and short keys

Recording a Macro: To create a macro, you start the Word macro recorder and record a sequence of actions, then stop the recorder and edit the macro if needed. You can then run the macro whenever you need to perform that same set of actions. The macro recorder can not record mouse actions in document text. You must use the key board when recording such actions as moving the insertion point and select text. To record a macro uses the following instructions:

- Select Tools and then Macro, then click on the Record New Macros button

  The Record Macro dialog box appears.

- Enter a name for the Macro in the Record Macro Name box. If you don't give your macro a name, Word will name it Macro 1, Macro 2 and so on. No spaces, commas, or period allowed in the name.
- Enter a description of what the macro does in the Description box. This is optional and you can use up to 255 characters.
- To assign a macro to a toolbar, a menu or a key board shortcut, click the appropriate button.
- If the current document is attached to a template other than the Normal template, select either that template or Normal in the Make Macro available to box. If the current document is attached to the Normal template the macro is automatically stored in the Normal template.
- Choose OK button, then perform the actions you want to record.
- To stop recording the macro, click the stop button on the Macro Recorder toolbar.

Running a Macro: Once you have recorded your macro, you can assign it to a toolbar, a menu or a shortcut key combination. You can then run it as you would a normal Word command or feature. If you don't want to assign a macro to a toolbar, or key combination, then you can also run a macro by choosing Tools → Macro, selecting the macro in the Macro dialog box, and clicking the Run button.

Editing and organizing a Macro: To edit a macro, do the following:

- Choose Macro from the Tools menu
- Select the macros again

  Macros dialog box is displayed.

- Select the macro you want to edit and then choose the Edit button.

Renaming a Macro: You can use the Organizer dialog box to manage your macro by renaming them. The steps to rename a macro are as follows:.

- Select Tools and then Macro

  The Macro dialog box will appear.
- Click the Organizer button

  The Organizer dialog box will appear with the Macros tab selected
- Choose the macro to rename
- Click the Rename botton
- Entering a new name for the macro and click OK

  The macro will be renamed.

Assigning Macros to a Menu, Toolbar and Short keys:

Assigning a macro to menu: To perform this following steps are required:

- Choose Customize from the Tools menu
- Select Commands
- Select Macros in the Categories box
- In the box to the right of the categories box, select the macro name to be assigned to a menu
- In the Change what menu box, select the name of the menu to which you want to assign the macro name
- In the position On Menu box, select the item below which you want to add the new item to position the item within the menu list

Assigning a Macro to a Toolbar button: For this follow the followings:

- Choose Customize from the Tools

  The Customize dialog box is appeard.
- Select the tool bars tab from the Customize dialog box
- In the Categories box, select Macros and choose the macro name to be assigned

- From the box to the right of the categories box, drag the macro name to the toolbar you want to add it to. As Macro don't have built-in-button, a blank button appears on the toolbar and customize button dialog box appears.
- In the button box, select an image to place an image on the blank button.
- Choose the assign button and then the Close button.

Assigning Macros to a Shortcut Key: The steps followed to assign to a shortcut key are as follows:

- Choose Customize from the Tools menu
- Select the key board tab
- In the Categories box, select Macros
- In the Box to the right of the Categories box, select the macro name to be assigned
- In the Press New Shortcut key box, type the shortcut key you want to assign to the macro
- Choose the assign button and close button

4.3.3 Auto Text

The Auto Text feature stores the text and graphics that are used repeatedly. It ensures that repetitive text is typed correctly and consistently. The steps to add text or graphics to an Auto Text entry are given below:

- Select the text or graphic items that you want to add to the Auto Text entry
- Click on the Auto Text option from the Insert menu.
- Type an abbreviated name for the text in the Name text box
- Click on the Add button

Delete Auto Text entry: The steps to delete an Auto Text entry are as follows:

- Select the Auto Text option from the Insert Menu
- Select the Auto Text name from the list shown or type the name of the Auto Text entry to be deleted in the Name text box
- Click on the Delete button.

4.3.4 Spell and Grammar Check

When a document is typed, no matter how careful the user is, mistakes are likely to be committed. Most of these are spelling errors. There are also some common grammatical mistakes. Word ensures that every document is free of spelling and grammatical errors. The steps to spell check and grammar are as follows:

- Select the section of the document to be spell-checked. If nothing is selected, Word checks the entire document.
- Select the Spelling and Grammar option from the Tools menu

  The Spelling and Grammar dialog box is displayed.
- Click on any one of the buttons explained below.

Button	Function
Change	Replaces the misspelled word
Change All	Replaces all occurrences of the misspelled word in the document
Ignore	Ignores the misspelled word
Ignore All	Ignores the word throughout the document
Add	Adds the word displayed in the Not in Dictionary text box to the dictionary
Undo Last	Reverse the last change
Cancel	Discontinues the spell check

A dialog box is displayed when the spell-check reaches the end of the document or the selection.

Word automatically underlines the misspelled words with a red wavy line. The steps to turn on automatic spell-check are as follows:

- Select the options item from the Tools menu

  The Options dialog box is displayed.
- Select the Spelling tab
- Select the Automatic Spell Checking check box

4.3.5 Auto Correct

The Auto Correct feature of Word recognizes common typing mistakes and automatically substitutes the correct spelling for the user. The steps to create entries for Auto Correct are as follows:

- Select the Auto Correct option from the Tools menu

  The Auto Correct dialog box is displayed.

- Type the misspelled word that is often misspelled and needs to be corrected automatically in the Replace text box.
- Type the correct spelling of the word in the With text box.
- Click on the Add button to add the new entry to the list of Auto Correct entries.
- Click on the OK button.

4.3.6 Protecting Document

The major file management operations include creating, opening, saving, restoring and protecting documents. Now the protecting a document is discussed below.

Protecting a document implies saving it from being changed either accidentally by some other user. A document can be protected in two ways:

- Opening a document as read only: When a document is opened as read-only the user can not make any changes to the document. This can be achieved by selecting the read-only check box in the Open dialog box. This read-only option will not allow any changes in the document to be saved.
- Protecting a document with a password: THe other way to protect a document is to use the Protection Password, which is typed in the Protection Password box. This can be done as follows:
  - Choose the Options from the Tools menu

    The Options dialog box is displayed.

  - Click on the Security tab
  - Type password name in the Password to open text box.
  - Click on OK button.

4.3.7 Using Find and Replace

The Find and Replace option of Word is a convenient way of searching way for a word in a document and replacing it with something else. The steps to find a specific text or special characters are as follows:

- Select the Find option from the Edit menu

  The Find and Replace dialog box is displayed

- Type the text to be searched in the Find what text box.
- Select one or more options explained below:

Option	Description
Search	Determines the direction of the search
Match Case	Matches the text exactly as it is typed, including capital letters
Find Whole Words Only	Finds whole words only, not parts of words
Use Pattern Matching	Use search operators and expressions with which to search
Sounds Like	Matches words that sound like
Find All Word Forms	Finds all forms of a word
Format	Displays the Format options including Font, Paragraph, Language and Style
Special	Enables the user to search for special codes in the text, such as tab characters
No Formatting	Removes any previous formatting

- Click on the Find Next button to begin the search.

The steps to replace text are as follows:

- Select the Replace option from the Edit menu
- Type the text to be replaced in the Find what text box
- Type the new text in the Replace with text box

- Select one or more options explained in above table
- Click on the Find Next button
- Click on the Replace button to change the text
- Click on the Cancel button to return to the document

4.3.8 Previewing a Document

Previewing a document meabs viewing a screen representation of one or more pages of the document before printing them. To print preview a document the following steps are required:

- Open the document
- From the File menu, choose the Print Preview option

  The Print Preview window is displayed and at the top of it there is a toolbar with buttons that perform the different actions.

- After viewing, click the Close button at the top of the Print Preview Window.

4.3.9 Printing a Document

The steps for printing a document are as follows:

- Connecting the Printer : Connect the printer directly to your computer or to a network.
- Select the Print option from the File menu or click on the Print. Toolbar on the standard toolbar.

  The Print dialog box is appeared.

- Select the printer on which you want to print the document in the Name text box
- Select one or more options explained below:

Option	Function
Print to file	Prints the document to a file on the disk
All	Prints all the pages of the document
Current Page	Prints only the current page of the document
Pages	Prints the range specified
Number of copies	Prints those many copies of the document, as specified
Collate	Prints a complete copy of the document before the first page of the next copy is printed

- Select an option from the Print what drop-down list
- Select an option from the Print drop-down list
- Click on the Properties button

  The Properties dialog box is displayed.
- Select the Paper tab
- Change the Paper size to A4 and Orientation to Portrait
- Click on the OK button to close the Properties dialog box
- Click on the OK button to send copies to the printer.

5.0 Unit Summary

In this module you learn the different techniques by which you can prepare a document completely and easily. In this module from the First Screen of MS-WORD to printing a document every things are discussed more elaborately.

6.0 Self Assessment Questions

1. Create a MS-WORD document with the following details:
   - Write a paragraph about your subject.
   - Create a table to store name, address and date of Birth, mobile number of your friend.
   - Insert footer in your document; write your name in it.

2. Create a data file containing Name, Address, and Mobile Number of your friends. Create a greeting of New Bengali Year. Use mailmerge feature to create the greeting letter foreach of your friends using the two files.

7.0 Suggested further Readings

1. Manual of MS-WORD
2. Books for IGNOU.

VIDYASAGAR UNIVERSITY

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M.Sc. in Applied Mathematics with Oceanology and Computer Programming

Part-I

Group-B Paper-V

Module No. - 57

COMPUTING METHODS

CONTENTS

1.0 Introduction

The adevent of high speed digital computers and its increasing use in finding numerical solution to problems of various branches of Science and Engineering has led to increased demand for courses on numerical analysis. Statistics is indispensable to different branches, e.g., planning, production control, etc. There are different techniques for the collection, presentation and analysis of the observed facts. In this unit the different methods relating to numerical and statistical are discussed.

2.0 Objectives

The working formula on Numerical Techniques, statistical techniques, concept of different searching and sorting techniques, rules for working in Turbo C and FORTRAN 77 are discussed in this unit.

3.0 Key Words and Study Guides

Bisection, Iteration, Newton-Raphson, mean, median, mode, binary search, linear search, quick sort, heap sort, merge sort.

4.0 Main Discussion

4.1 WORKING FORMULA ON NUMERICAL TECHNIQUES

4.1.1 Method of Bisection

This method is based on the theorem which state that if a function $f[x]$ is continuous between *a* and *b*, and $f(a)$ and $f(b)$ are of opposite signs, there exists at least one root between *a* and *b*. Let $f(a)$ and $f(b)$ be of opposite signs. Then the root lies between *a* and *b* and its approximate value be given by $x_{mid} = \frac{a+b}{2}$. Now if $f(x_{mid}) = 0$, we conclude that x_{mid} is a root of the equation $f(x) = 0$. Otherwise, the root lies either between *a* and x_{mid}, or between x_{mid} and *b* depending on whether $f(x_{mid})$ is negative or positive. If $f(a)f(x_{mid}) < 0$, the root lies between (a, x_{mid}) and if $f(x_{mid})f(b) < 0$ then the root lies between (x_{mid}, b). Then we bisect the interval and repeat the process until the root is known to the desired accuracy.

Algorithm:

Step-1: Read *a*, *b* and ε where (a,b) is the initial interval and ε is the desired accuracy.

Step-2: Compute $y_0 = f(a)$ and $y_1 = f(b)$.

Step[-3: Check if $y_0 y_1 > 0$ then go to Step-1.

Step-4: Compute $x_{mid} = \frac{(a+b)}{2}$.

Step-5: Check if $f(x_{mid}) = 0$ then go to Step-8.

Step-6: Check if $f(a) f(x_{mid}) < 0$ then set $b = x_{mid}$.

Else $a = x_{mid}$

Step-7: Check if $|a-b| > \varepsilon$ then go to Step-4.

Step-8: Print the root x_{mid}.

Step-9: Stop.

Flowchart: The flowchart of this algorithm is given on the next page.

4.1.2 Method of Iteration

In this method the equation $f(x) = 0$ is re-written in the form $x = \phi(x)$ such that $\phi(x)$ is continuous in the interval that contains the root the equation and $|\phi'(x)| < 1$ in this interval. Let x_0 be an approximate value of the desired root. We then substitute this value of x in $\phi(x)$ and we get the first approximation and the successive approximations as

$$x_1 = \phi(x_0)$$

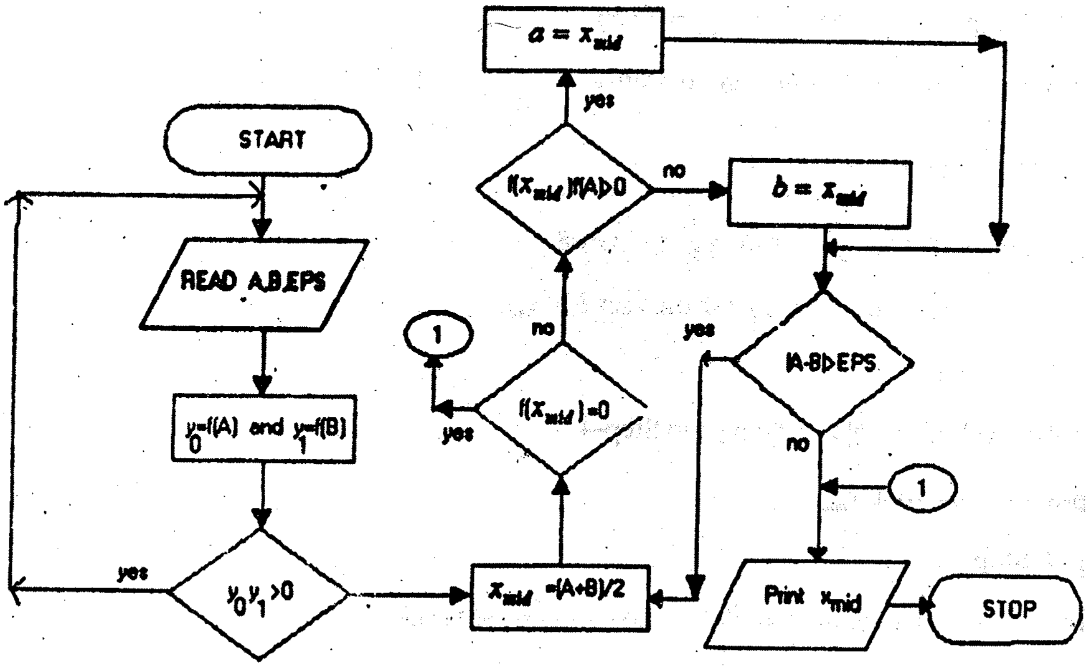
$$x_2 = \phi(x_1)$$

....

....

$$x_{n+1} = \phi(x_n)$$

The sequence $x_0, x_1, x_2, ..., x_n, x_{n+1}, ...$ will converge to a root.



Altorithm:

Step-1: Read x_0, ε and n. Here, x_0 is the initial guess of the root, ε is allowable error, n is maximum number of iterations.

Step-2: Set $i \leftarrow 0$

Step-3: Compute $x_1 = \phi(x_0)$

Step-4: Compute $i = i + 1$

Step-5: Check if $|x_1 - x_0| < \varepsilon$ then go to Step-9

Step-6: Check if $i > n$ then go to step-11

Step-7: Set $x_0 \leftarrow x_1$

Step-8: Go to Step-3

Step-9: Print the root x_1

Step-10: Go to 12

Step-11: Print, "method fails"

Step-12: Stop

4.1.3 Regula-Flasi Method

In this method, it requires to know the approximate location of the root in the given interval in which a real root lies. Let at the points $a = x_0$ and $b = x_1$, the values of the function $f(x_0)$ and $f(x_1)$ are of opposite signs, so that one root must lie between these two points x_0 and x_1. Now we replace the curve $y = f(x)$ by the equation of the chord joining the points $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$ as

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

The point where the above chord cuts the x-axis is obtained by putting y=0. Therefore the next approximation we get

$$x_2 = x_0 - \frac{f(x_0)}{x_1 - x_0}(x_1 - x_0)$$

Now if $f(x_0)$ and $f(x_2)$ are of opposite signs then the root lies between x_0 and x_2 otherwise the root lies in the interval x_2 and x_1. Then this process is repeated until the root is obtained to the required degree of accuracy.

Algorithm:

Step-1: Read n, ε. Here ε is allowable error.

Step-2: Set $i \leftarrow 0$

Step-3: Read x_1, x_2. Here, x_1, x_2 are two initial guesses.

Step-4: Compute $f_1 = f(x_1)$ and $f_2 = f(x_2)$

Step-5: Compute $i = i + 1$

Step-6: Check if $f_1 \times f_2 > 0$ and $i \leq n$ then go to Step-3.

Step-7: Check if $i > n$ then go to Step-13.

Step-7: Check if $|x_1 - x_2| < \varepsilon$ then go to Step-9

Step-8: Compute $x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

Step-9: Check if $f(x_3)=0$ then go to Step-

Step-10: Check if $f(x_1)f(x_3)<0$ then

$x_2 \leftarrow x_3$

Else

$x_1 \leftarrow x_3$

Step-11: Go to Step-7

Step-12: Print the root x_3.

Step-13: Go to Step-15.

Step-14: Print, "There is no real root"

Step-15: Stop

4.1.4 Newton Raphson Method

To find a simple (non-repeated) real root of the equation $f(x)=0$, we may use the Newton-Raphson method. Let x_0 be an approximate value fo the desired root. Then by this method the successive approximations of the required root are $x_1, x_2, x_3, ..., x_{n+1}$ given by the followings

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

.........................,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Algorithm:

Step-1: Read x_0, ε, n. Here x_0 is the initial guess of the root, ε is the desired accuracy and n is the maximum number of iterations.

Step-2: Set $i \leftarrow 0$

Step-3: Compute $f_0 = f(x_0)$

Step-4: Compute $f_0' = f'(x_0)$

Step-5: Check if $f_0' = 0$ then Print, "Method fails" and Go to Step-12

Step-6: Compute $x_1 = x_0 - \frac{f_0}{f_0'}$

Step-7: Compute $i = i + 1$

Step-8: Check if $|x_1 - x_0| < \varepsilon$ Print, the root x_1 and go to Step-12

Step-9: Set $x_0 \leftarrow x_1$

Step-10: Check $i \le n$ then go to step-3.

Step-11: Print, "Method fails"

Step-12: Stop

4.1.5 Roots of Polynomial Equation by Quotient-difference method

This is a general method to obtain the approximate roots of polynomial equations. Let the polynomial be given by

$$f(x) = a_1 x^{n-2} + \ldots\ldots + a_n$$

and let $x_1, x_2, \ldots\ldots, x_n$ be the simple roots of $f(x) = 0$. A computational procedure by this method is described as follows:

$$d_0^k = d_n^k = 0, k = 0,1,2\ldots\ldots$$

$$d_i^0 = \frac{a_{i+1}}{a_i}, i = 1,2,\ldots\ldots, n-1$$

$$q_i^0 = \frac{a_1}{a_0}, q_i^0 = 0, i = 2,3\ldots\ldots, n$$

$$q_i^k = q_i^{k-1} + d_{i-1}^{k-1}, i = 1,2,\ldots, n \text{ and } k = 1,2,\ldots\ldots$$

$$d_i^k = \frac{q_{i+1}^k \times d_i^{k-1}}{q_i^k}, i = 1,2,\ldots\ldots, n-1 \text{ and } k = 1,2,\ldots\ldots$$

Convergence criteria: If a_i's for $i = 0,1,2,...n$ are non-zero and all roots are simple with $0 < |x_1| < |x_2| < |x_3| < ...... < |x_n|$, then when $\lim_{n\to\infty} d_i^n = 0$, then $\lim_{n\to\infty} q_i^n = x_i \forall n$

Algorithm:

Step-1: Read n which is the degree of a polynomial.

Step-2: Read all coefficients $a_0, a_1, ..., a_n$

Step-3: Read the maximum number of iteration it_{max}

Step-4: Set $k \leftarrow 0$

Step-5: $q_1 = -a_1 / a_0$

Step-6: Do for $i = 2$ to n

Step-7: $q_i = 0.0$

Step-8: End do

Step-9: Set $d_0 \leftarrow 0, d_n \leftarrow 0$

Step-10: Do for $i = 1$ to $n - 1$

Step-11: Compute $d_i = a_{i+1} / a_i$

Step-12: End do

Step-13: Set $id \leftarrow 0$

Step-14: Do for $i = 1$ to $n - 1$

Step-15: if $|d_i| < 0.001$, then do

id = 1

else

id=0

goto Step-17

Step-16: End do

Step-17: if it is equal to 1, then go to step-25.

Step-18: Do for i - 1 to n

Step-19: Compute $q_1 = q_1 + d_1 - d_{i-1}$

Step-20: End do

Step-21: do for $i = 1$ to $n - 1$

Step-22: Compute $d_i = q_{i+1} * d_i / q_i$

Step-23: End do

Step-24: if $k < it_{max}$ then goto Step-13.

Step-25: Print, "the roots are", $q_i, i = 1, \ldots\ldots, n$

Step-26: Print, "the iteration:", k

Step-27: Stop.

4.1.6 Solution of Tri-diagonal system

We consider the system of equations defined by

$$
\begin{aligned}
b_1x_1 + c_1x_2 \quad &= d_1 \\
a_2x_1 + b_2x_2 + c_2x_3 \quad &= d_2 \\
+a_3x_2 + b_3x_3 + c_3x_4 \quad &= d_3 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
a_nx_{n-1} + b_xx_n \quad &= d_n
\end{aligned}
$$

This is a tri-diagonal system if the coefficient matrix is positive definite. A computational procedure due to Thomas, is described as follows:

The coefficient matrix is converted to the upper bi-diagonal form as below

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & b_2^* & c_2^* & 0 & \cdots\cdots & 0 & 0 \\ 0 & 0 & b_3^* & c_3^* & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & b_{n-1}^* & c_{b-1}^* \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 & b_n^* \end{bmatrix}$$

and the requirement vector $d^* = \left[d_1 d_2^* d_3^* ......d_n^*\right]$. Hence solving the above system we have as follows

$$x_n = \frac{d_n^*}{b_n^*}, x_i = \frac{d_i^* - c_i x_{i+1}}{b_i^*}, i = n-1, n-2, ..., 2, 1$$

where $c_i^* = c_i - \frac{a_i c_{i-1}}{b_{i-1}^*}$ and $d_i^* = d_i - \frac{d_{i-1} a_i}{b_{i-1}^*}, i = 2, 3, ..., n$

Algorithms:

Step-1: Read n, n is the order of the coefficient matrix.

Step-2: read the coefficient matrix $A_{ij}, i, j = 1, 2, ..., n$.

Step-3: Read the requirement vector $B_i, i = 1, 2, ..., n$.

Step-4: Do for i=2 to n

Step-5: Compute $A_{i,i} = A_{i,i} - A_{i,i-1} * A_{i-1,i} / A_{i-1,i-1}$

$B_i = B_i - B_{i-1} * A_{i-1,i} / A_{i-1,i-1}$.

Step-6: End do

Step-7: Compute $x_i = B_n / A_{n,n}$

Step-8: Do for i=n-1 to 1 decreasing by -1

Step-9: Compute $x_i = \left(B_i - A_{i,i+1}\right) / A_{i,i}$

Step-10: End do

Step-11: Print the values of $x_i = i = 1, 2, ..., n$ which is the required solution.

Step-12: Stop.

4.1.7 Boundary value problems by Finite difference method

A boundary value problem of a second-degree ordinary differential equation is defined as

$$\frac{d^2y}{dx^2}+f(x)\frac{dy}{dx}+g(x)y=r(x)$$

with the boundary conditions

$$\left.\begin{array}{l} y(x_0)=a \\ y(x_n)=b \end{array}\right\}$$

There exist many methods of solving second order boundary value problem. Of these the finite-differnce method is a popular one, which is described as follows:

The finite-difference method for the solution of a two-point boundary value problem consists in replacing the derivatives occurring in the differential equation by means of their finite-difference approximations and then solving the resulting linear system of equations by a standard procedure. In this method the finite differences for $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ are as follows:

$$y''(x)=\frac{y(x-h)-2y(x)+y(x+h)}{h^2}+O(h^2) \text{ and}$$

$$y'(x)=\frac{y(x+h)-y(x-h)}{2h}+O(h^2)$$

Process: To solve the boundary value problem defined above, we divide the range $[x_0, x_n]$ into n equal subintervals of width h so that

$$x_i = x_0 + ih, i = 1,2,3,......,n$$

The corresponding values of y at these points are denoted by

$$y(x_i)=y_i=y(x_0+ih), i=0,1,2,......,n$$

From the equations above, values of $y'(x)$ and $y''(x)$ at the point $x=x_i$ can now be written as

$$y'_i=\frac{y_{i+1}-y_{i-1}}{2h}+O(h^2)$$

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + O(h^2)$$

Solving the differential equation at the point $x = x_i$, we get

$$y_i'' + f_i y_i' + g_i y_i = r_i$$

Substituting the expressions for y_i' and y_i'', this gives

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + f_i \frac{y_{i+1} - y_{i-1}}{2h} + g.y_i = r_i, i = 1, 2, ..., n-1$$

where $y_i = y(x_i), f_i = f(x_i), g_i = g(x_i)$ and $r_i = r(x_i)$.

Multiplying through by h^2 and simplifying, we obtain

$$\left(1 - \frac{h}{2} f_i\right) y_{i-1} + \left(-2 + g_i h^2\right) y_i + \left(1 + \frac{h}{2} f_i\right) y_{i+1} = r_i h^2, i = 1, 2, ..., n-1$$

with $y_0 = a, y_n = b$.

These equations with the conditions comprise a tri-diagonal system, which can be solved by the method discussed above.

4.2 WORKING FORMULA ON STATISTICAL TECHNIQUES

Mean: Arithmetic mean of a set of observations is their sum divided by the number of observations i.e., the arithmetic mean $\bar{x}$ of n observations $x_1, x_2, ..., x_n$ is given by the following:

$$\bar{x} = \frac{1}{x}(x_1 + x_2 + ... + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_1$$

In case of the frequency distribution $(x_i, f_i, i = 1, 2, ..., n)$ where f_i is the frequency of the variable x_i, the arithmetic mean is given by

$$x = \frac{f_1 x_1 + f_2 x_2 + ... + f_n x_n}{f_1 + f_2 + ... + f_n} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

Algorithm:

Step-1: Read the size of frequency distribution, n

Step-2: Read the frequency distribution x_i, f_i for $i = 1, 2, ..., n$

Step-3: Set $i \leftarrow 1, mean \leftarrow 1, mean \leftarrow 0$ and $N \leftarrow 0$

Step-4: Check if $i > n$ then go to step-8

Step-5: Compute $mean = mean + f_i x_i$ and $N = N + f_i$

Step-6: Compute $i = i + 1$

Step-7: Goto Step-4

Step-8: Compute $mean = \frac{mean}{N}$

Step-9: Print, the mean $mean$

Step-10: Stop.

Median: Median of a distribution is the value of the variable, which divides it into two equal parts. It is the value such that the number of observations above it is equal to the number of observations below it. The median is thus a positional average.

In case of ungrouped data, if the number of observations is odd then median is the middle value after the values have been arranged in ascending or descending order of magnitude. In case of even number of observations, there are two middle terms and median is obtained by taking the arithmetic mean of the middle terms.

In case of grouped data, the class corresponding to the cumulative frequency just greater than half of total frequency is called the **median class** and the value of median is obtained by the following formula:

$$Median = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

where l is the lower limit of the median class,

f is the frequency of the median class,

h is the magnitude of the median class,

c is the cumulative frequency of the class preceding the median class, and

$$N = \sum_{i=1}^{n} f_i$$

Mode: Mode is the value, which occurs most frequently in a set of observations and around which the other items of the set cluster densely. In other words, mode is the value of the variable, which is predominant in the series. Therefore, **in the case of discrete frequency distribution,** mode is the value of x corresponding to maximum frequency and in case of continuous frequency distribution; mode is given by the following formula:

$$Mode = l + \frac{h(f_1 - f_0)}{(f_1 - f_0) - (f_2 - f_1)}$$

where l is the lower limit of the modal class,

h is the magnitude of the modal class,

f_1 is the frequency of the modal class,

f_0 and f_2 are the frequencies of the class preceding and succeeding the modal class.

Standard deviation: Standard deviation, usually denoted by σ, is the positive squaer root of the arithmetic mean of the squares of the deviations of the given values from their arithmetic mean. For the frequency distribution $(x, f_i, i = 1, 2, ..., n)$, the standard deviation is given by

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \bar{x})^2}$$

where $\bar{x}$ is the arithmetic mean of the distribution and $N = \sum_{i=1}^{n} f_i$

Correlation coefficient: In a bivariate distribution we may be interested to find out if there is any correlation or covariance between the two variables under study. If the change is one variable affects a change in the other variable, the variables are said to be correlated. If the two variables deviate in the same direction, i.e. if the increase (or decrease) in one result in a corresponding increase (or decrease) in the other, correlation is said to be direct or **positive.** But if they constantly deviate in the opposite directions, i.e., if increase (or decrease) in one results in corresponding decrease (or increase) in the other, correlation is said to be diverse or **negative.** For example, the correlation between (i) the heights and weights of a group of persons, and (ii) the income and expenditure is positive and the correlation between (i) price and demand of a commodity and (ii) the volume and the pressure of a perfect gas, is negative. Correlation is said to be perfect if the deviation in one variable is followed by a corresponding and proportional deviation in the other.

As a measure of intensity or degree of linear relationship between two variables, Karl Pearson developed a formula called Correlation Coefficient. Correlation coefficient between two random variables *X* and *Y*, usually denoted by *r(X, Y)* or simply *r* is a numerical measure of linear relationship between them and it is defined by

$$r = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \overline{xy}}{\sqrt{\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2 - \bar{y}^2\right)}}$$

Curve fitting: To fit a straight line through a set of points $(x_i, y_i), i = 1, 2, \ldots, n$, the straight line equation is

$$y = mx + c \text{ where } m = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_1^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \text{ and } c = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i\right)$$

4.3 METHODS ON SEARCHING AND SHORTING

Bubble Sort: Bubble sort is a simple sorting technique. Adjacent elements are compared. If they are not in order they are swapped. For example, while sorting in ascending order, the element with the largest value will move to the end of the array in the first phase. After the first pass, since the element with the largest has reached its final position, the size of the array to be sorted reduces by 1. This process goes on till all elements are in their respective positions.

Algorithm:

Step-1: Read n, n is the number of elements to be sorted.

Step-2: Read all elements A_i for $i = 1, 2, \ldots, n$

Step-3: Set $i \leftarrow 1$

Step-4: Set $j \leftarrow 1$

Step-5: Check if $A_j > A_{j+1}$ then go to Step-6, otherwise go to Step-9

Step-6: Set $temp \leftarrow A_i$

Step-7: Set $A_j \leftarrow A_{j+1}$

Step-8: Set $A_{j+1} \leftarrow temp$

Step-9: Compute $j = j + 1$

Step-10: Check if $j \leq n - i$ go to Step-5

Step-11: Compute $i = i + 1$

Step-12: Check if $i \leq n - 1$ then go to Step-4

Step-13: Print, the sorted elements $A_1, A_2, \ldots, A_n$

Step-14: Stop

Insertion Sort: The basic step in this method is to insert an element into a sequence of ordered elements $A_1, A_2, A_3, \ldots, A_i$ $(A_1 \leq A_2 \leq A_3 \leq \ldots \leq A_i)$ in such a way that the resulting sequence of size $i + 1$ is also ordered. The following algorithm accomplished this insertion.

Altorithm:

Step-1: Read n where n is the number of elements.

Step-2: Read allelements $A_1, A_2, A_3, \ldots, A_n$

Step-3: Set $j \leftarrow 2$

Step-4: Check if $j > n$ then go to Step-13

Step-5: Set $key \leftarrow A_j$

Step-6: Set $i \leftarrow j-1$

Step-7: while $i > 0$ and $A_i > key$

Step-8: Set $A_{i+1} \leftarrow A_i$

Step-9: Compute $i = i - 1$

Step-10: end while

Step-10: Set $A_{i+1} \leftarrow key$

Step-11: Computer $j = j + 1$

Step-12: Go to Step-4

Step-13: Print, A_k, for $k = 1, 2, ..., n$

Step-14: Stop.

Merge Sort: We assume that the elements are to be sorted in non-decreasing order. Given a sequence of n elements $A_1, A_2, ..., A_n$, the general idea is to imagine them split into two set $A_1, ..., A_{\left[\frac{n}{2}\right]}$ and $A_{\left[\frac{n}{2}+1\right]}, ..., A_n$. Each set is individually sorted, and the resulting sorted sequences are merged to produce a single sorted sequence of n elements.

The algorithm is given below as a function form and the initial calling of the main function is MERGESORT $(1, n)$.

Algorithm:

Step-1: Read n

Step-2: Read $A_1, A_2 ......, A_n$

Step-3: Call the function *MERGESORT*$(1, n)$

Step-4: Print, the sorted elements $A_1, A_2 ......, A_n$.

Step-5: Stop.

MERGESORT (low, high)

if $(low < high)$ *then*

$$mid = \left[\frac{low + high}{2}\right]$$

MERGESORT (low, mid)

MERGESORT (mid+1, high)

MERGE (low, mid, high)

end if

End MERGESORT

MERGE (low, mid, high)

Set $h \leftarrow low$

Set $i \leftarrow low$

Set $j \leftarrow mid + 1$

while $((h \le mid)$ and $(j \ge high))$ do

Check if $A_h \le A_j$ then

$B_i \leftarrow A_h$

$h = h + 1$

else

$B_i \leftarrow A_j$

$j = j + 1$

end if

calculate $i = i + 1$

end while

Check if $h > mid$ then

for $k = j$ *to high* do

Set $B_i \leftarrow A_k$

Calculate $i = i + 1$

Else

For $k = h$ *to mid do*

Set $B_i \leftarrow A_k$

Calculate $i = i + 1$

End if

For k = *low to high do*

Set $A_k \leftarrow B_k$

End Merge.

Example: We consider the array of ten elements A[1..10]={310, 285, 179, 652, 351, 423, 861, 254, 450, 520}. The algorithm MERGE_SORT begins by spliting A[] into two subarrays each of size five **(A[1...5] and A[6...10]).** The

in A[1...5] are then split into two subarrays of size three (A[1...3]) and two **(A[4...5]).** Then the items in A[1...3] are split into subarrays of size two **(A[1...2])** and one **(A[3...3]).** The two valves in A[1...2] are split a final time into one-element subarrays and now the merging begings. We note that no movement of data has yet taken place. Pictorially the file can now be viewed as

(310 | 285 | 179 | 652, 351 | 423, 861, 254, 450, 520)

where vertical bars indicate the boundaries of subarrays. Now A[1] and A[2] are merged to yield **(285, 310 | 179 | 652, 351 | 423, 861, 254, 450, 520).** Then A[3] is merged with A[1...2] and **(178, 285, 310 | 652, 351 | 423, 861, 254, 450, 520)** is produced. Next elements A[4] and A[5] are merged and we get

(179, 285, 310 | 351, 652 | 423, 861, 254, 450, 520)

and then A[1...3] and A[4...5] are merged and we get

(179, 285, 310, 351, 652 | 423, 861, 254, 450, 520)

At this point the algorithm has returned to the first invocation of MERGE_SORT and is about to process the second recursive call. Repeated recursive calls are invoked producing the following subarrays:

(179, 285, 310, 351, 652 | 423, 861, 254, 450, 520)

The elements A[6] and A[7] are merged. Then A[8] is merged with A[6...7].

(179, 285, 310, 351, 652 | 254, 423, 861 | 450, 520)

Next A[9] and A[10] are merged and then A[6...8] and A[9...10]:

(179, 285, 310, 351, 652 | 254, 423, 450, 520, 861)

At this point there are two sorted subarrays and the final merge produces the full sorted result.

(179, 285, 310, 351, 652 | 254, 423, 450, 520, 861)

Quick Sort: Quick Sort is based on the a[vide- and-conquer paradigm, which is the three-step procedure for sorting a sub array $A_p, A_{p+1}, ..., A_r$ discussed below:

Divide: The array $A_p, A_{p+1}, ..., A_r$ is partitioned into two non-empty sub arrays $A_p, A_{p+1}, ..., A_q$ and $A_{q+1}, A_{p+1}, ..., A_r$ such that each element of $A_p, A_{p+1}, ..., A_q$ is less than or equal to each element of $A_{q+1}, A_{p+1}, ..., A_r$. The index q is computed as part of this partitioning procedure.

Conquer: The two sub arrays $A_p, A_{p+1}, ..., A_q$ and $A_{q+1}, A_{p+1}, ..., A_r$ are sorted recursive calls to quick sort.

Combine: Since the sub arrays are sorted in place, no work is needed to combine them: the entire array $A_p, A_{p+1}, ..., A_r$ is now sorted.

The following procedure implements quick sort.

```
QUICKSORT (A,p,r)
if p < r then
q ← PARTITION (A,p,r)
QUICKSORT (A,p,q)
QUICKSORT (A,q+1,r)
End QUICKSORT
```

Partitioning the array: The key to the algorithm is the *PARTITION* procedure, which rearranges the sub array $A_p, A_{p+1}, ..., A_r$ in place. Conceptually, the partitioning procedure performs a simple function: it puts elements smaller than x into the bottom region of the array and elements larger than x into the top region.

PARTITION(A,p,r)

$x \leftarrow A_p$

$i \leftarrow p-1$

$j \leftarrow r+1$

while TRUE

do

repeat $j \leftarrow j-1$

until $A_i \geq x$

if $i < j$ *then*

exchange $A_i \leftrightarrow A_j$

else

return j

End PARTITION

Heapsort: The (binary) heap data structure is an array object that can be viewed as a complete binary tree as shown in the following figure. Each node of the tree corresponds to an element of the array that stores the value in the node. The tree is completely filled on all leaves except possibly the lowest, which is filled from the left up to a point. An array A that represents a heap is an object with two attributes: *length* $[A]$, which is the number of elements in the array, and *heapsize*$[A]$, the number of elements in the heap stored within array A, where *heapsize*$[A] \leq$ *length*$[A]$. The root of the tree is $A[1]$, and given index i of a node, the indices of its parent *PARENT*(i), left child *LEFT*(i), and right child *RIGHT*(i) can be computed as follows:

PARENT (i)

return $[i/2]$

LEFT (i)

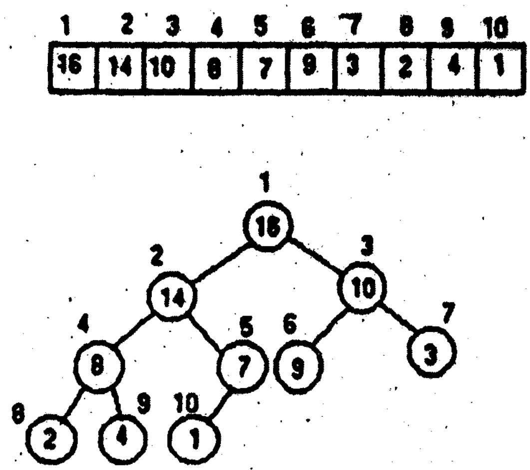
return $2i$

RIGHT (i)

return $2i+1$

Heap also satisfies the heap property: for every node i other than the root, $A[PARENT(i)] \geq A[i]$

Example:



The algorithm *HEAPIFY* maintains the heap property. Its inputs are an array A and an index i into the array. If $A[i]$ is smaller than its children, then it violets the heap property. The function *HEAPIFY* is to let the value at $A[i]$ float down in the heap so that the subtree rooted at index i becomes a heap.

HEAPIFY (A,i)

$l \leftarrow LEFT(i)$

$r \leftarrow RIGHT(i)$

if $l \leq heapsize[A]$ *and* $A[l] > A[i]$ ***then***

largest $\leftarrow l$

else

largest $\leftarrow i$

if $r \leq heapsize[A]$ *and* $A[r] > A[largest]$ ***then***

largest $\leftarrow r$

if largest $\neq i$ *then*

exchange $A[i] \leftrightarrow A[largest]$

HEAPIFY (A, largest)

End HEAPIFY

We can use the procedure *HEAPIFY* in a bottom-up manner to convert an array $A_1, A_2, ...A_n$ where n = *length*[*A*], into a heap. Since the elements in the sub array $A_{\lfloor n/2 \rfloor+1}, ...A_n$ are all leaves of the tree, each is a 1-element heap to begin with. The following procedure *BUILD HEAP* goes through the remaining nodes of the tree and runs *HEAPIFY* on each one.

BUILD HEAP(A)

$heapsize[A] \leftarrow length[A]$

for i $\lceil length[A]/2 \rceil$ *downto* 1

do $HEAPIFY(A,i)$

End BUILDHEAP

The heap sort algorithm starts by using *BUILDHEAP* to build a heap on the input array $A_1, A_2, ...A_n$ where $n = length[A]$. Since the maximum element of the array is sorted at the root $A[1]$, it can be put into its correct final position by exchanging it with $A[n]$. If we now discard node n from the heap (by decrementing $heapsize[A]$), we observe that $A_1, A_2, ...A_{n-1}$ can be easily made into a heap. The children of the root remain heaps, but the new root element may violet property. All that is needed to restore the heap property, however, is one call to $HEAPIFY(A,1)$, which leaves a heap in $A_1, A_2, ...A_{n-1}$. The following heap sort algorithm then repeats this process for the heap of size $n - 1$ down to a heap of size 2.

HEAPSORT(A)

BUILDHEAP(A)

for i ← *length* [*A*] *downto 2*

do

exchange A [1] ↔ *A*[*i*]

heapsize $[A]$ *heapsize* $[A]-1$

HEAPIFY$(A,1)$

End BUILDHEAP

Binary Search: We consider that $A[i], i=1,2,.....n$, be a list of elements that are sorted in non-decreasing order. The problem of determining whether a given element x is present in the list. If x is present, we are to determine a value j such that $A[i]=x$. If x is not present in the list, then j is to be set zero. We say that $P=(n, A[i], A[i+1],......, A[l], x)$ denote an arbitrary instance of this search problem. Divide-and-conquer can be used to solve this problem. We say that *Small(P)* is a Boolean-valued function that determines whether the input size is small enough that the answer can be computed without splitting. If this is so, the function *S* is invoked. Otherwise the problem *P* is divided into smaller subproblems. In binary search, *Small(P)* be true if n=1. In this case, *S(P)* will take the value : *if* $x=A[i]$, otherwise it will take the value 0. If *P* has more than one element, it can be divided (or reduced) into a new subproblem as follows. Pick an index q (in the range $[i,1]$) and compare x with $A[q]$. There ar three possibilities:

1. $x=A[q]$: in this case the problem *P* is immediately solved.
2. $x<A[q]$: in this case x has to be searched for only in the sublist $A[i.q-1]$.

   Therefore, p reduces to $(q-i, A[i], A[i+1],......, A[q-1], x)$.
3. $X>A[q]$: in this case the sublist to be searched is $A[q+1...1]$. P reduces to $(l-q, A[q+1], A[q+2],......, A[l], x)$.

Algorithm:

BINSRCH (A, i, l, x)

//Given an array $A[i.l]$ of elements in nondecrea sin g order,

//$1\le i\le$, determine e whether x is present and if so, return j such that

//$x=A[j]$, else return 0.

if (l=i) then // if Small (P)

```
if (x = A [i]) then return i;
else return 0;
else
{// reduce P into a smaller subproblem
mid = |_(i+1)/2_|;
if (x = A[mid]) then return mid;
else if (x < A[mid]) then
        return BINSRCH (A, i, mid - 1, x);
    else
        return BINSRCH (A, mid + 1, i, x);
}
```

Random Number Generator: A commonly used method to generate random numbers is the linear congruential method. In this method, each number r_k in the sequences of random numbers is calculated from its predecessor r_{k-1} using the following formula

$$r_k = (multiplier \times r_{k-1} + increment)\%\ modulus$$

where multiplier, increment, and modulus are appropriate chosen constants. The sequence generated by this formula is really pseudo-random, since the value of r_k can always be predicted, given r_0.

4.4 EXECUTING TURBO C

4.4.1 Introduction: All programs in C are run under Turbo C complier. We discuss briefly the creation and execution of C programs under Turbo C system.

4.4.2 Creation and Execution of Programs: Executing a computer program written in any high-level language involves several steps as follows:

a) Develop the program (source code)

b) Select a suitable file name under which you would like to store the program.

c) Create the program in the computer and save it under the filename you have decided. This file is known as **source code file.**

d) Compile the source code. The file containing the translated code is called **object code file.** If there are any errors, debug them and compile the program again.

e) Link the object code with other library codes that are required for execution. The resulting code is called the **executable code.** If there are errors in linking, correct them and compile the program again.

f) Run the executable code and obtain the results, if there are no errors.

g) Debus the program, if errors are found in the output.

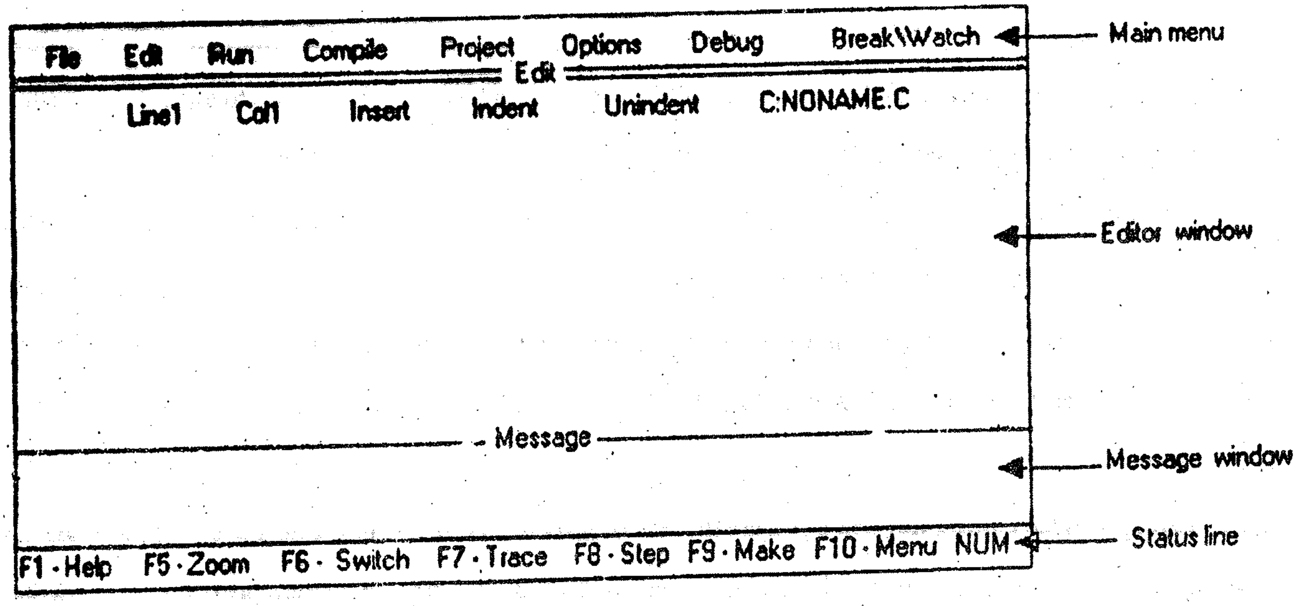
h) Go to step d and repeat the process again.

The exact steps depend upon the program environment and the compiler used. But, they will resemble the steps described above. Turbo C is the most popular C compiler. It provides ideal platform for learning and developing C programs. Therefore we discuss here about Turbo C.

4.4.2 Turbo C: It provides a powerful environment called Integrated Development Environment (IDE) for creating and executing a program. It is completely menu-driven and allows the user to create, edit, compile and run programs. Single keystrokes and easy-to-use menus control these operations. We first use the editor to create the source code file, then compile and finally run it. Turbo C provides error messages, in case errors are detected. We have to correct the errors and compile the program again.

4.4.3 IDE Screen: It is important to be familiar with the details of the IDE screen that will be extensively used in the program development and execution. When we invoke the Turbo C, the IDE screen will be displayed as below. This screen has four parts:

- Main menu (top line)
- Editor window
- Message window
- Status line (bottom line)



Main Menu: The main menu lists a number of items that are required for the program development and execution. These are summarized as below:

Item	Action
File	Loads and saves files, handles directories, invokes DOS, and exit Turbo C.
Edit	Performs various editing functions.
Run	Compiles, links, and runs the program currently loaded in the environment.
Compile	Compiles the program currently in the environment.
Project	Manages multi-file projects.
Option	Sets various compiler, linker, and environmental options.
Debug	Sets various debugger option.
Break/Watch	

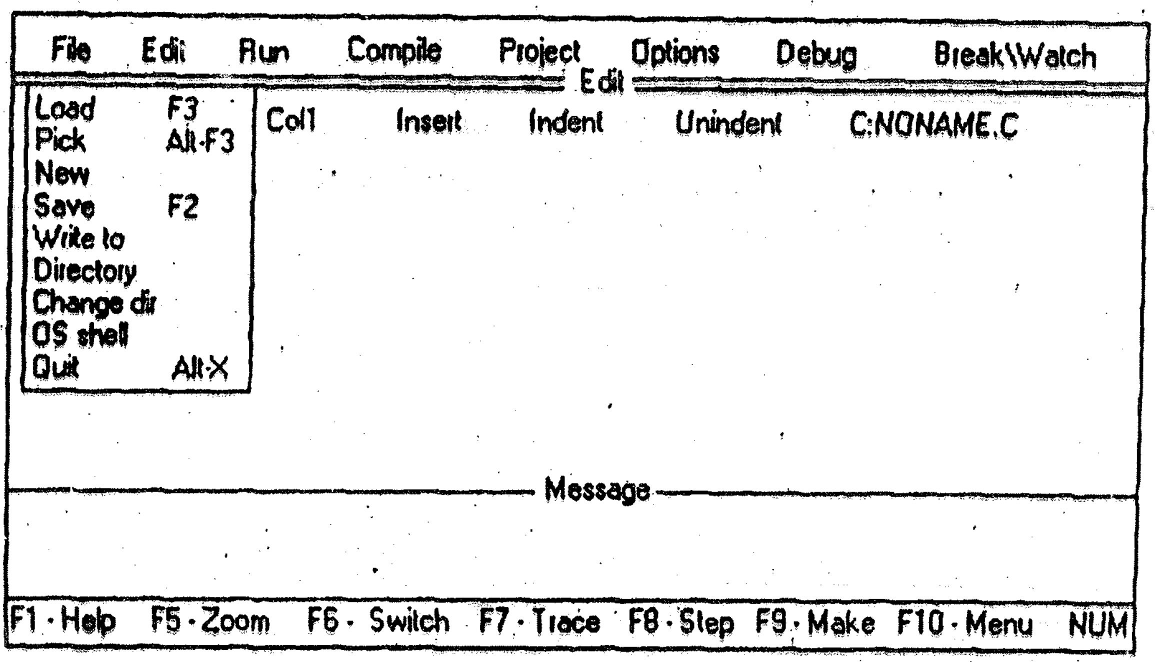
Pressing the F10 key can activate the main menu: When we select an item on the main menu, a pull-down menu, containing various options, is displayed. This allows us to select an action that relates to the main menu item.

Editor Window: The editor window is the place for creating the source code of C program. This window is named NONAME00.C. This is the temporary name given to a file, which can be changed while we save the file.

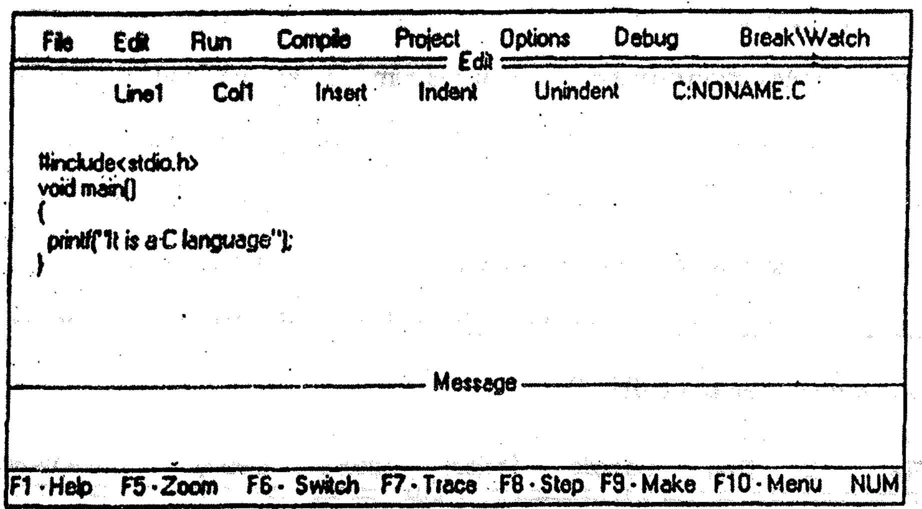
Message Window: The other window on the screen is called the watch window where various messages are displayed. The messages may be compiler and linker messages and error messages generated by the compiler.

Status Line: The status line, which is displayed at the bottom of the screen, gives the status of the current activity on the screen.

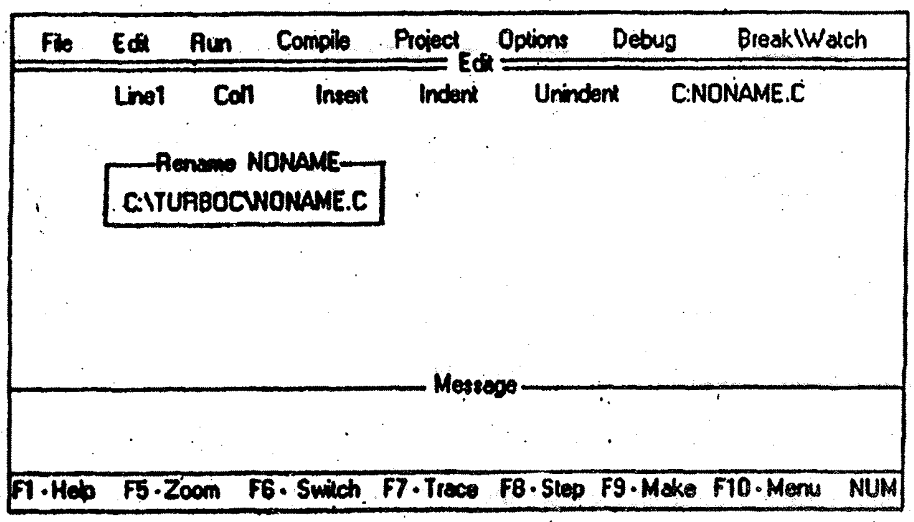
4.4.4 Creating Source Code File: Once you are in the IDE screen, it is simple to create and save a program. The F10 key will take you to main menu and then move the curson to *File*. On the *File* item press down-arrow-key. This will display the file dialogue window containing various options for file operations as shown below. The options include, among others, opening an existing file, creating a new file and saving the new file.



Since you want to create a new file, move the cursor to *New* option. This opens up a blank window called editing window and places the cursor inside this window. Now the system is ready to receive the program statements as shown below



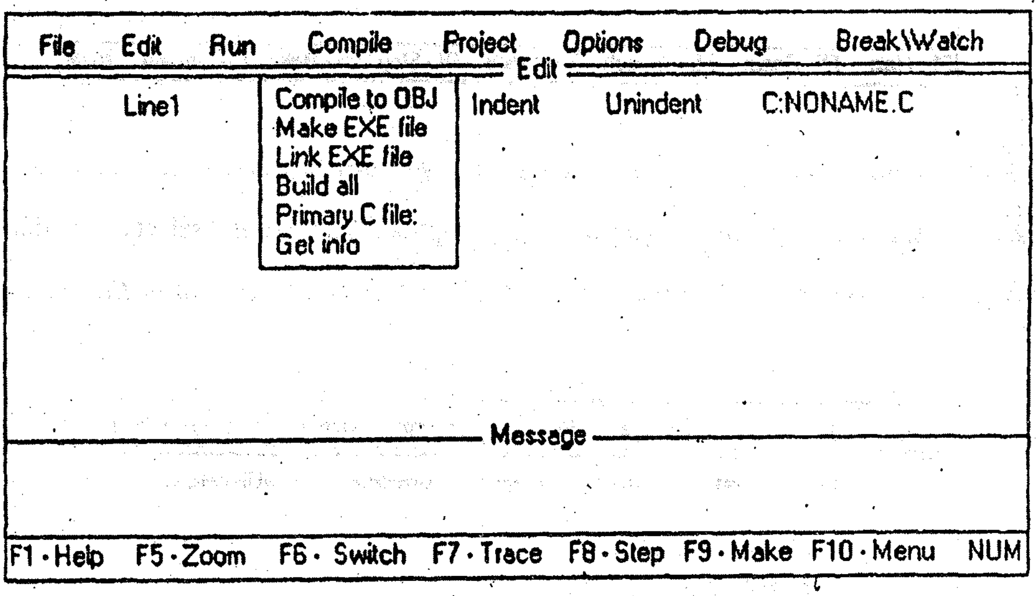
4.4.5 Saving Source Code File: Once the typing is completed, you are ready to save the program in a file. At this time, you must go to the File dialogue menu again to select file option. Press F10 and select *File* option on the main menu. Press down-arrow-key and select the *save* option. This brings the *save editor file window* as shown below:



Now, you may change the file name *NONAME.C* (shown in the editor file window) to name you have selected. Make sure that your name has the extension *.C* to indicate to the compiler that your program is a *C* one. Assume that you have selected *TEST.C* as name. Now press *RETURN* key and the program is saved in the file *TEST.C*.

4.4.6 Compiling the program:

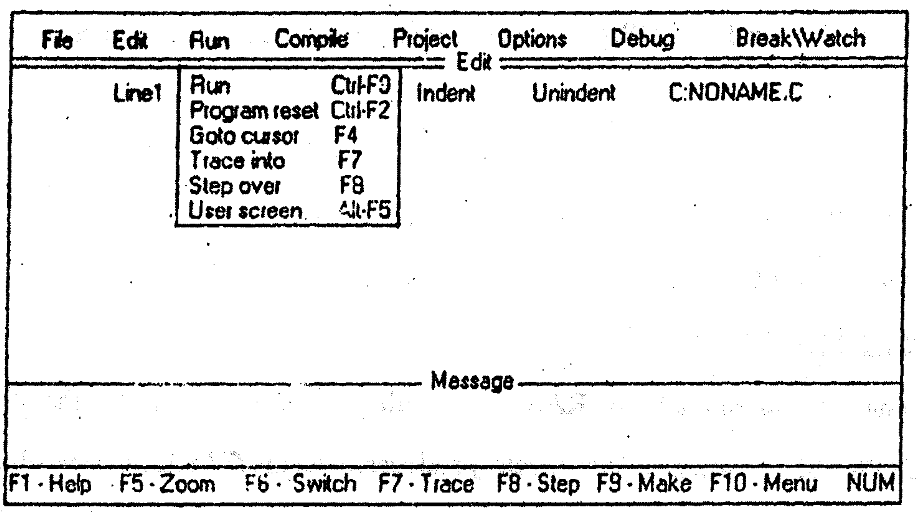
When you select the compile option on the main menu, the compile dialogue window is displayed as shown below. The *compile to OBJ* option allows you to compile the current file in the editor to an object file. In the present case, *TEST.OBJ* file is created, if there are no errors in your program. If there are any errors, appropriate error messages are displayed in the message window. During compilation, a window called *compilation window* will appear on the screen. If there are no errors during compilation, this window will display **"Success: Press any key"** message. The entries for *warning and errors* will be 0.



4.4.7 Running the program:

You have reached successfully the final stage of your excitement. Now, select the **Run** option from the main menu and again **Run** from the *run dialogue window* as shown below. You will see the screen flicker briefly. Surprisingly, no output is displayed. Where has the output gone? It has gone to a place known as *user screen*. In order to see the user screen, select **Run** option from the main menu and then select *user screen* from the *run dialogue menu*.

The IDE screen will disappear and the *user screen* is displayed containing output of the program TEST.C. At this point, you are outside the IDE. To return to IDE, press RETURN key. You can also toggles between the *user screen* and IDE.



4.4.8 Managing Errors:

It is rare that a program runs successfully the first time. It is common to make some syntax errors while preparing the program or during typing. Fortunately, the compiler or linker detects all such errors, which are discussed as follows:

Compiler Errors: The compiler will detect all syntax errors. For example, if you have missed the semicolon at the end of the printf-statement in TEST.C program, the message about this will be displayed in the message window. By pressing ENTER you can go to *Edit window* that contains your program. Correct the errors and then compile and run the program again.

Linker Errors: It is also possible to have errors during the linking process. For example, you may not have include the file *stdio.h.* The program will compile correctly, but will fail to link. It will display an error message in the linking window. Press any key to see the message in the message window.

Run-time Errors: Remember that compiling and linking successfully do not always guaranty the correct results. Sometimes, the results may not be wrong to **logical errors** or due to errors such as **stack overflow**. System might display the errors such as null pointer assignment.

4.4.9 Invoking Turbo c:

Assume that you have installed the Turbo C compiler correctly and suppose that it is in the directory *C:\TURBOC*. In WINDOWS 95/98, you go to that directory and search the file *TC.EXE* and then double click this file. After that IDE screen will be appeared. In DOS, you go to that directory in which the compiler is loaded. Then enter *tc* at the DOS system prompt:

C:\TURBOC > tc

and then press RETURN key. This will place you into the IDE screen.

4.5 EXECUTING FORTRAN-77

4.5.1 Invoking FORTRAN-77:

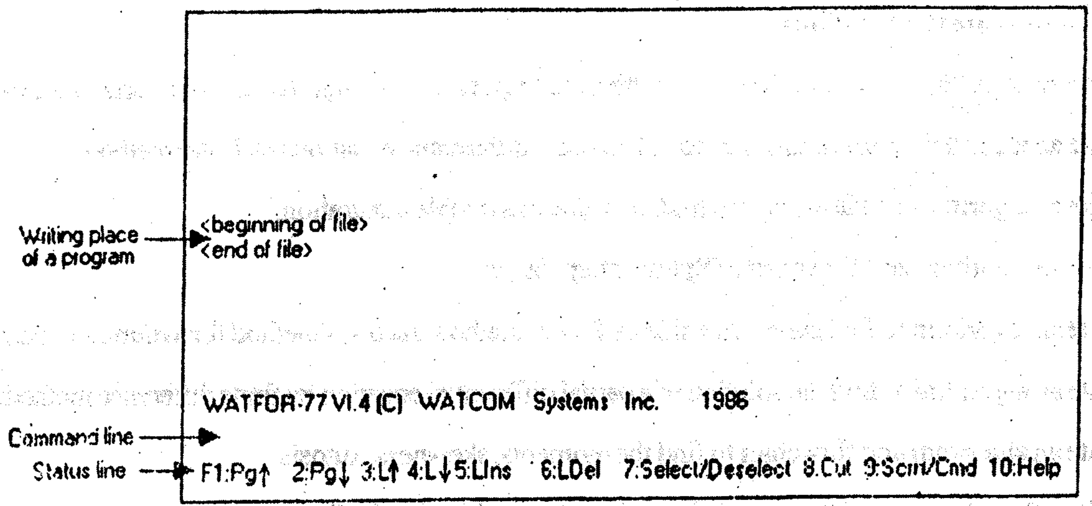
Assume that you have installed the FORTRAN-77 compiler correctly and suppose that it is in the directory *C:\F77*. In WINDOWS 95/98, you go to that directory and search the file *f77.EXE* and then double click this file. After that Editor screen will be appeared. In DOS, you go to that directory in which the compiler is loaded. Then enter *F77* at the DOS system prompt:

C:\F77 > F77

and then press RETURN key. This will place you into the Editor screen.

4.5.2 Editor Screen:

The first screen after invoking F77 in which place a program have to be written and the output will shown as below. There is three parts: (i) Writing place of a program, (ii) Command line, and (iii) Status line. Now to switch over the cursor from the Command line to < *beginning of file* > and vice-versa, we have to press *F9* key in the keyboard. To write a program we require the line spacing. But we see that there is no line in between of <*beginning of file*> and <*end of file*>. So to write a program, first we require the line gap in between these.



For line insertion, press the key F5. For example if we want to create three lines, then we have to press the key F5 three times. In this way after the insertion of lines, we write a program according to FORTRAN-77 coding rules, which is shown as a sample program.

4.5.3 Main Commands to manage FORTRAN-77:

Commands	Functions
PUT filename	It saves the program to the filename
RUN filename	It executes the file containing the program
EDIT filename	It opens the file filename and also creates a file by this name
F5	It insert a blank line
F6	It delete a line
F9	It switches over from command line to writing place of the program and vice versa

5.0 Unit Summary

In this module, working formula on Numerical Techniques, statistical techniques, concept of different searching and sorting techniques, rules for working in Turbo C and FORTRAN 77 are discussed.

6.0 Self Assessment Questions

1. Write an algorithm to solve a first order differential equation by Runge-Kutta fourth order method.
2. Write an algorithm to solve simultaneous first order differential equations by Euler method.
3. Write an algorithm and flowchart to find the value of a double integration.
4. Write an algorithm and flowchart of Spline interpolation.
5. Write an algorithm to find the eigen value by Power method, Jacobi's method for symmetric matrix.
6. Write an algorithm to find the solution of a partial differential equation by finite difference method.
7. Write an algorithm and flowchart to find the moments, skewness, kutosis.
8. Write a flowchart to find the value of mean, median and mode of a discrete distribution.
9. Write an algorithm for curve fitting.
10. Write an algorithm to prepare a frequency distribution.

7.0 Suggested further Readings

1. Datta, N. and Jana, R. Introductory Numerical Analysis.
2. Sastry, S.S., Introductory Methods of Numerical Analysis.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-V **Group-B**

Module No. - 58
PROGRAMMING IN C-IV

Content :

MODULE 58 :Programming in C

1.0 Introduction

In less than a decade of its introduction, C has become a langauge of choice for software professionals. Numerical analysis is concerened with methods, which give numerical solution to mathematical problems by artihmetic operations on numbers. In Applied Mathematics, Theoretical Physics and Engineering, the ultimate problem is to computer numerical results using certain data. With the advent of modern high-speed digital computers the numerical analysis of today is a very different discipline. To implement numerical methods C programming is very useful to day.

2.0 Objectives

In this module the following topics are implemented :

* Solution of algebraic and transcendental equations
* Solution of a system of linear equations
* Numerical Integration
* Interpolation
* Numerical solution of ordinary differential equation
* Eigen value problem.

3.0 Key Words and Study guides

Interpolation, Integration, Linera equations, Algebraic equation, transcendental equation.

4.0 Numerical Problems

4.1.1. SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

4.1.1.1 Write a program in C to find a real root of an equation by Bisection method. Test the program taking the example $f(x) = x^3 - x - 1 = 0$.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

#define MAXCOUNT 10

void main ()
```

```
{
float a,b,x0,xmid;
int count=0;
float f(float);
do{
clrscr();
printf("Enter the interval(a,b) in which a root exits: ");
scanf("%f%f",&a,&b);
}while((f(a)*f(b)>0)&&(count++<MAXCOUNT));
xmid=(a+b)/2.0;
do{
x0=xmid;
if(f(x0)= =0)
        break;
else if(f(a)*f(x0)<0)
        b=x0;
else
        a=x0;
xmid=(a+b)/2.0;
}while(fabs(xmid-x0)>0.0001);
printf("The required root is %f and f(%f)=%f",x0,x0,f(x0));
}
float f(float x)
{
float val;
val=pow(x,3)-x-1;
return val;
}
```

4.1.1.2 Write a program in C to find a real root of an equation by Iteration method. Test the program taking the example $2x = \cos x + 3$.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

#define MAXCOUNT 10

void main ()
```

```
{
float a,b,x0,x1;
int count=0;
float f(float);
float phi (float);
do{
clrscr();
printf("Enter the interval(a,b) in which a root exits: ");
scanf("%f%f",&a,&b);
}while((f(a)*f(b)>0&&(count++<MAXCOUNT));
x1=a;
do{
x0=x1;
x1=phi(x0);
}while(fabs(x1-x0)>0.0001);
printf("The required root is %f and f(%f)=%f",x1,x1,f(x1));
}
float f(float x)
{
float val;
val=2*x-cos(x)-3;
return val;
}
float phi (float x)
{
float val;
val=(cos(x)+3)/2.0;
return val;
}
```

4.1.1.3. Write a program in C to find a real root of an equation by Regula-Falsi method. Test the program taking the example $f(x)=x^3-2x-5=0$.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

#define MAXCOUNT 10

void main ()
```

```
{
float x0,x1,x2new;
int count=0;
float f(float);
do{
clrscr();
printf("Enter the interval(a,b) in which a root exits: ");
scanf("%f'f",&x0,&x1);
}while((f(x0)*f(x1)>0)&&(count++<MAXCOUNT));
x2new=x0-f(x0)/(f(x1)-f(x0))*(x1-x0);
do{
x2=x2new;
if(f(x2)==0)
        break;
else if(f(x0)*f(x2)<0)
        x1=x2;
else
        x0=x2;
x2new=x0-f(x0)/(f(x1)-f(x0))*(x1-x0);
}while(fabs(x2new-x2)>0.0001);
printf("The required root is %f and f(%f)=%f",x2,x2,f(x2));
}
float f(float x)
{
float val;
val=pow(x,3)-2*x-5;
return val;
}
```

4.1.1.4 Write a program in C to find a real root of an equation by Newton-Raphson's method. Test the program taking the example $x \sin x + \cos x = 0$.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

#define MAXCOUNT 10

void main ()
```

```
{
float a,b,x0,x1,fdx0;
int count=0;
float f(float);
float fd(float);
do{
clrscr();
printf("Enter the interval(a,b) in which a root exits: ");
scanf("%f%f",&a,&b);
}while((f(a)*f(b)>0)&&(count++<MAXCOUNT));
x1=a;
do{
x0=x1;
fdx0=fd(x0);
if(fdx0= =0)
{
printf("The method fails");
exit(1);
}
x1=x0-f(x0)/fdx0;
}while(fabs(x1-x0)>0.0001);
printf("The required root is %f and f(%f)=%f",x1,x1,f(x1));
}
float f(float x)
{
float val;
val=x*sin(x)+cos(x);
return val;
}
float fd(float x)
{
float val;
val=x*cos(x);
return val;
}
```

4.1.2. SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

4.1.2.1 Write a program in C to find all roots of a system of linear equations by Gauss Elimination method. Test using the following:

$$2x + y + z = 10,\ 3x + 2y + 3z = 18,\ x + 4y + 9z = 16$$

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

#define SIZE 5

void main ()
{
float a[SIZE][SIZE+1],x[SIZE];
float ratio;
int n,i,j,k;
clrscr();
printf("Enter the order of the co-efficient matrix: ");
scanf("%d",&n);
printf("Enter the co-efficients of Augmented matrix row-wise:\n");
for(i=0;i<n;i++)
for(j=0,j<n+1,j++)
scanf("%f",&a[i][j]);

for (k=0;k<n-1;k++)
for(i=k+1,i<n;i++)
{
ratio=a[i][k]/a[k][k];
for(j=0;j<n;j++)
a[i][j]=a[i][j]-ratio*a[k][j];
}
x[n-1]=a[n-1][n]/a[n-1][n-1];
```

```
for(k=n-2;k>=0;k--)
{
x[k]=a[k][n];
for(j=k+1,j<n,j++)
x[k]=x[k]-a[k][j]*x[j];
x[k]=x[k]/a[k][k];
}
printf("The Required Solution is/n");
for(i=0,i<n;i++)
printif("x[%d]=%f\t",i+1,x[i]);
}
```

1.2.2 Write a program in C to find all roots of a system of linear equations by Gauss Seidal method. Test using the lloiwng:

$10x - 2y - z - w = 3, - 2x + 10y - z - w = 15,$

$-x - y + 10z - 2w = 27, -x - y - 2z + 10w = -9$

```
#include<stdio.h>
#include<conio.h>
#include<stdlib.h>
#include<math.h>

#define SIZE 5

void main ()
{
float a[SIZE][SIZE+1],abssum,xn[SIZE],xo[SIZE];
int n,i,j,iteration,id;
clrscr();
```

```
printf("Enter the order of the co-efficient matrix: ");
scanf("%d",&n);
printf("Enter the all co-efficients of Augmented matrix row-wise:\n");
for(i=0,i<n;1++)
for(j=0;j<n+1,j++)
scanf("%f",&a[i][j]);
for(i=0,i<n;i++)
{
abssum=0.0;
for(j=0;j<n;j++)
if(i==j)
        continue;
else
        abssum=abssum+fabs(a[i][j]/a[i][i]);
if(abssum>1.0)
{
        printf("The converging condition is not satisfied");
        exit(1);
}
{
iteration=0;
for(i=0;i<n;i++)
{
xn[i]0.0;
xo[i]=0.0;
}
do{
iteration=iteration+1;
```

```
for(i=0,i<n;i++)
{
xn[i]=a[i][n];
for(j=0,j<n;j++)
if(i==j)
        continue;
else
        xn[i]=xn[i]-a[i][j]*xn[j];
xn[i]=xn[i]/a[i][i];
}
id=0;
for(i=0,i<n;i++)
if(fabs(xn[i]-xo[i]>0.00001)
        id=1;
if(id==1)
for(j=0;j<n;j++)
        xo[j]=xn[j];
else
break;
}while(1);
printf("The required solution\n");
for(i=0,i<n;i++)
printf("x[%d]=%f/t",i+1,xn[i]);
printf("\nat the iteration %d", iteration);
}
```

4.1.2.3 Write a program in C to find all roots of a system of linear equations by Matrix Inversion method. Test using the following:

$2x + y + z = 10,\ 3x + 2y + 3z = 18,\ x + 4y + 9z = 16$

```
#include<stdio.h>
#include<conio.h>
#include<stdlib.h>
#include<math.h>

#define SIZE 5

void main ()
{
float a[SIZE][SIZE], ia[SIZE][SIZE], B[SIZE], X[SIZE];
float ratio, det;
int n,i,j.k;
float determinant(float a[SIZE][SIZE], int n);
clrscr();
printf("Enter the order of the co-efficient matrix: ");
scanf("%d",&n);
printf("Enter the co-efficient matrix:\n");
for(i=0,i<n;i++)
for(j=0,j<n,j++)
scanf("%f",&a[i][j]);
printf("Enter the Requirement Vector:\n");
for(i=0,i<n;i++)
scanf("%f",&b[i]);
det=determinant(a,n);
if(det= =0)
{
```

```
printf("Solution does not exist");
exit(1);
}
for(i=0,i<n;i++)
for(j=0,j<n;j++)
it(i==j)
ia[i][j]=1.0;
else
ia[i][j]=0.0;

for(k=0;k<n;k++)
for(i=0;i<n;i++)
{
if(i==k)
continue;
else
ratio=a[i][k]/a[k][k];
for(j=0,j<n;j++)
{
	a[i][j]=a[i][j]-ratio*a[k][j];
	ia[i][j]=ia[i][j]-ratio*is[k][j];
}
}
for(i=0;i<n;i++)
for(j=0;j<j;j++)
ia[i][j]=ia[i][j]/a[i][i];
for(i=0,i<n;i++)
{
```

```
x[i]=0.0;
for(j=0;j<n;j++)
x[i]=x[i]+ia[i][j]*b[j];
}
printf("The solution is \n");
for(i=0;i<n;i++)
printf("x[%d]=%f\t",i+1,x[i]);
}
float determinant(float a[SIZE][SIZE], int n)
{
float det, ratio;
int i,j,k;
for(k=0;k<n-1;k++)
for(i=k+1;i<n;i++)
{
ratio=a[i][k]/a[k][k];
for(j=0;j<n;j++)
a[i][j]=a[i][j]-ratio*a[k][j];
}
det=1.0;
for(i=0;i<n;i++)
det=det*a[i][i];
return det;
}
```

4.2.3 INTERPOLATION

4.1.3.1 Write a program in C to form a difference table using the following data.

x	0	1	2	3
y	1	0	1	10

```
#include<stdio.h>
#include<conio.h>

#define SIZE 10

void main ()
{
float x[SIZE],yd[SIZE][SIZE];
int i,j,n,rowvalue,columvalue;
clrscr();
printf("Enter No. of points:");
scanf("%d",&n);
printf("Enter x and y values \n");
for(i=0;i<n;i++)
scanf("%f%f",&x[i]&yd[i][0]);
rowvalue=n;
for(j=0;j<n-1;j++)
{
rowvalue=rowvalue-1;
for(i=0;i<rowvalue;i++)
yed[i][j+1]=yd[i+1][j]-yd[i][j];
}
printf("\n***The Difference Tabl***\n");
columvalue=n;
for(i=0;i<n;i++)
{
columvalue=columvalue-1;
for(j=0;j<=columvalue;j++)
printf("%0.1f/t",yd[i][j]);
printf("\n");
}
}
```

4.1.3.2 Write a program in C to find a value of a function by Lagrange Interpolation Technique. Using this to find the following : evaluate $\sqrt{155}$ with the following table

x	150	152	154	156
$y = \sqrt{x}$	12.247	12.329	12.410	12.490

```
#include<stdio.h>
#include<conio.h>

#define SIZE 10

void main ()
{
float xf,x[SIZE],y[SIZE], term. sumvalue;
int i,j,n;
clrscr();
printif("Enter total no. of points (x,y):");
scanf("%d",&n);
printf("Enter all points (x,y);\n");
for(i=0;i<n;i++)
scanf("%f%f",&x[i],&y[i]);
printif("Enter x-value for f(x) calculation: ");
scanf("%f"<&xf);
sumvalue=0.0;
for(i=0;i<n;i++)
{
term=y[i];
for(j=0;j<n;j++)
if(i!=j)
term=term*(xf-x[j])/(x[i]-x[j]);
sumvalue=sumvalue+term;
}
printf("The required value: f(%f)=%f",xf,sumvalue);
getch();
}
```

4.1.3.3 Write a program in C to find a value of a function by Newton Forward Interpolation Technique. Using this to find the following : evaluate sin(0.175) with the following table

x	0.15	0.17	0.18	0.21	0.23
sin x	0.14944	0.16918	0.18886	0.20846	0.22798

```
#include<stdio.h>
#include<conio.h>

#define SIZE 10

void main ()
{
float xf,x0,y0,x[SIZE],y[SIZE],h,s,term,sumvalue;
int i,j,n,totalterm,x0index;
clrscr();
printf("Enter total no. of points (x,y):");
scanf(%d",&n);
printf("Enter all points (x,y):\n");
for(i=0;i<n;i++)
scanf("%f%f",&x[i],&y[i]);
printf("Enter x-value for f(x) calculation:");
scanf("%f",&xf);
for(i=1,i<n-1;i++)
if(xf<x[i])
{
x0=x[i-1];
y0=y[i-1];
x0index=i-1;
totalterm=n+1-i;
break;
}
h=x[1]-x[0];
s=(xf-x0)/h;
sumvalue=y0;
term=1.0;
for(i=1;i<=totalterm-1;i++)
{
for(j=0;j<n-i;j++)
y[j]=y[j+1]-y[j];
term=term*(s-i+1)/i;
sumvalue=sumvalue+term*y[x0index];
}
printf("The required value : f(%f)=%f",xf,sumvalue);
getch();
}
```

4.1.3.4 Writea program in C to find a value of a function by Newton Backward Interpolation Technique. Using this to find the following : evaluate $\sqrt{155}$ with the following table

x	150	152	154	156
$y=\sqrt{x}$	12.247	12.329	12.410	12.490

```
#include<stdio.h>
#include<conio.h>

#define SIZE 10

void main ()
{
float xf, x[SIZE],y[SIZE],h,s,term,sumvalue;
int i,j,j;
clrscr();
printf("Enter total no. of points (x,y):");
scanf("%d",&n);
printf("Enter all points (x,y);\n");
for(i=0;i<n;i++)
scanf("%f%f",&x[i],&y[i]);
printf("Enter x-value for f(x) calculaton:");
scanf("%of",&xf);
h=x[n-1]-x[n-2];
s=(xf-x[n-1])/h;
sumvalue=y[n-1];
term=1.0;
for(i=1;i<n-1;i++)
{
for(j=0;j<n-i,j++)
y[j]=y[j+1]-y[j];
term=term*(s+i-1)/i;
sumvalue=sumvalue+term*y[n-i-1];
}
printf("The required value:f(%f)=%f",xf,sumvalue);
getch();
}
```

4.1.4 INTEGRATION

4.1.4.1 Write a program in C to find the value of $\int_a^b f(x)\,dx$ by Trapezoidal Rule. Using the following example test the program: $\int_0^1 \frac{1.0}{1.0+x}dx$

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

void main ()
{
float a,b,h,xi,sum=0.0;
int n,i;
float f(float);
printf("Enter the lower and upper limites of the integrations:");
scanf("f%f",&a,&b);
printf("Enter no. of subinterval:");
scanf("%d",&n);
h=(b-a)/n;
for(i=1,i<n-1,i++)
{
xi=a+i*h;
sum=sum+2.0*f(xi);
}
sum=sum+f(a)+f(b);
sum=h/2.0*sum;
printf("The value of the Integration is %f", sum);
}
float f(float x)
{
float val;
val=1.0/(1.0+x);
return val;
}
```

4.1.4.2 Write a program in C to find the value of $\int_a^b f(x)\,dx$ by Simpson's 1/3-Rule. Using the following example test the program: $\int_0^1 \frac{1.0}{1.0+x}dx$

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

void main ()
{
float a,b,h,xi,sum=0.0;
int n,i;
float f(float);
clrscr();
printf("Enter the lower and upper limits of the integration:");
scanf('%f%f',&a,&b);
printf("Enter no. of subinterval:");
scanf("%d",&n);
if(n= =(n/2)*2)
{
h=(b-a)/n;
for(i=1;i<=n-1,i=i+2)
{
xi=a+i*h;
sum=sum+4.0*f(xi);
}
for(i=2,i<n-2;i=i+2)
{
xi=a+i*h;
sum=sum+2.0*f(xi);
}
sum=sum+f(a)+f(b);
sum=h/3.0*sum;
printf("The value of the Integration is %f",sum);
}
else
printf("For Simson's 1/3, the no. of sub-intervals must be even");
}
float (f(float x)
```

```
{
float val;
val=1.0/(1.0+x);
return val;
}
```

4.1.4.3 Write a program in C to find the value of $\int_a^b f(x)\,dx$ by Simpson's 3/8-Rule. Using the following example test the program: $\int_0^1 \frac{1.0}{1.0+x}dx$

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
#include<stdlib.h>

void main ()
{
float a,b,h,xi,sum=0.0;
int n,i,sindex,eindex,tcount;
float f(float);
clrscr();
printf("Enter the lower and upper limits of the integration:");
scanf("%f%f",&a,&b);
printf("Enter no. of subinterval:");
scanf("d",&n);
if(n!=(n/3)*3)
{
printf("Please enter no. of subinterval which is multiple of three");
exit(1);
}
h=(b-a)/n;
sindex=0;
eindex=3;
while(einde<=n)
{
tcount=0;
for(i=sindex;i<=eindex;i++)
{
xi=a+i*h;
tcount=tcount+1;
```

```
if(tcount==2|| tcount==3)
      sum=sum+3.0*f(xi);
else
      sum=sum+f(xi);
}
sindex=eindex;
eindex=eindex+3;
}
sum=3.0*h/8.0*sum;
printf("The value f the Integration is %f",sum);
}
float f(float x)
{
float val;
val=1.0/(1.0+x);
return val;
}
```

.4 Write a program in C to find the value of $\int_a^b f(x)\,dx$ by Weddles Rule. Using the following example

the program: $\int_0^1 \frac{1.0}{1.0+x}dx$

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
#include<stdlib.h>

void main ()
{
float a,b,h,xi,sum=0.0;
int n,i,sindex,eindex,tcount;
float f(float);
clrscr();
printf("Enter the lower and upper limits of the integration:");
scanf("f%f",&a,&b);
printf("Enter no. of subinterval:");
scanf("%d",&n);
if(n!=(n/6)*6)
{
printf("Please enter no. of subinterval which is multiple of six");
```

```
exit(1);
}
h(b-a)/n;
sindex=0;
eindex=6;
while(eindex<=n)
{
tcount=0;
for(i=sindex;i<=eindex;i++)
{
xi=a+i*h;
tcount=tcount+1;
if(tcount= =2 || tcount= =6)
sum=sum+5.0*f(xi);
else if(tcount= =4)
sum=sum+6.0*f(xi);
else
sum=sum+f(xi);
}
sindex=eindex;
eindex=eindex+6;
}
sum=3.0*h/10.0*sum;
printf("The value of the Integration is %f",sum);
}
float f(float x)
{
float val;
val=1.0/(1.0+x);
return val;
}
```

4.1.4.5 Write a program in C to find the value of $\iint_{a\le x\le b, c\le y\le x} f(x,y)\,dxdy$ by Weddles Rule. Using the following example test the program: $\iint_{0\le x\le 1, 0\le y\le 1} e^{x+y}\,dxdy$.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
#include<stdlib.h>
```

```
void main ()
{
float a,b,c,d,h,k,sum=0.0;
int n,m,i,j;
float f(float,float);
clrscr();
printf("Enter the x-lower and x-upper limits of the integration:");
scanf("f%f",&a,&b);
printf("Enter the y-lower and y-upper limits of the integration:");
scanf("%f%f",&c,&d);
printf("Enter no. of x-subinterval:");
scanf("d",&m);
printf("Enter no. of y-subinterval:");
scanf("%d",&n);
h=(b-a)/m;
k=(d-c)/n;
for(i=0;i<m;i++)
for(j=0;j<n;j++)
sum=sum+f(a+i*h,c+j*k)+f(a+(i+1)*h,c+j*k)
       +f(a+i*h,c+(j+1)*k)+f(a+(i+1)*h,c+(j+1)*k);
sum=h*k/4.0*sum;
printf("The value of the Double Integration is %f",sum);
}
float f(float x,float y)
{
float val;
val=exp(x+y);
return val;
}
```

4.1.4.6 Write a program in C to find the value of $\iint_{a\le x\le b,\, c\le y\le x} f(x,y)\,dxdy$ by Simpson's Rule. Using the following example test the program: $\iint_{0\le x\le 1,\, 0\le y\le 1} e^{x+y}\,dxdy$.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
#include<stdlib.h>

void main ()
```

```
{
float a,b,c,d,h,k,sum=0.0;
int n,m,i,j;
float f(float,float);
clrscr();
printf("Enter the x-lower and x-upper limits of the integration:");
scanf("f%f",&a,&b);
printf("Enter the y-lower and y-upper limits of the integration:");
scanf("%f%f",&c,&d);
printf("Enter no. of x-subinterval:");
scanf("d",&m);
printf("Enter no. of y-subinterval:");
scanf("%d",&n);
h=(b-a)/m;
k=(d-c)/n;
for(i=1;i<m;i++)
for(j=1;j<n;j++)
sum=sum+f(a+(i-1)*h,c+(j-1)*k)+f(a+(i-1)*h,c+(j+1)*k)+f(a+(i+1)*h,c+(j-1)*k
     +f(a+(i+1)*h,c+(j+1)*k)+4*(f(a+(i-1)*h,c+j*k)+f(a+i*h,c+(j-1)*k)
     +f(a+i*h,c+(j+1)*k)+f(a+(i+1)*h,c+j*k))+16*f(a+i*h,c+j*k);
sum=h*k/9.0*sum;
printf("The value of the Double Integration is %f",sum);
}
float f(float x,float y)
{
float val;
val=exp(x+y);
return val;
}
```

4.1.5 SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

4.1.5.1 Write a program in C to solve a differential equation by Euler's method and take the foll example. Given $\frac{dy}{dx} = y - x$, where $y(0) = 2$, find $y(0.2)$.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
```

```
void main ()
{
float h,x,x1,y1,x0,y0;
float f(float, float);
clrscr();
printf("Enter the step length h: ");
scanf("%f",&h);
printf(Enter the initial x and y values:");
scanf("%f'f",&x0,&y0;
printf("Enter the x-value at which y can be determined: "):
scanf("%f",&x);
printf("-----------------------------/n");
printf("      x          y/n");
printf("----------------------------/n");
while(x0<x)
{
x1=x0+h;
y1=y0+h*(f(x0,y0);
printf("     %f          %f/n",x1,y1);
x0=x1;
y0=y1;
}
printf("------------------------/n");
printf(Hence y(%f)=%f",x1,y1);
}
float f(float x, float y)
{
float val;
val=y-x;
return val;
}
```

4.1.5.2 Write a program in C to solve a differential equation by Modified Euler's method and take the following example. Given $\frac{dy}{dx} = y - x$, where $y(0) = 2$, find $y(0.2)$.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
```

```
void main ()
{
float h,x,x1,y1,x0,y0,x1n,y1n,y1n_old;
float f(float,float);
clrscr();
printf("Enter the step length h:");
scanf("f",&h);
printf("Enter the initial x and y values: ");
scanf("%f%f",&x0,&y0);
printf("Enter the x-value at which y can be determined: ");
scanf("%f",&x);
printf("-----------------------------/n");
printf("      x         y/n");
printf("---------------------------/n");
y1=y0+h*f(x0,y0);
while(x0<x)
{
x1=x0+h;
y1n=y1;
do{
y1n_old=y1n;
y1n=y0+h/2.0*(f(x0,y0)+f(x1,y1n));
}while(fabs(y1n-y1n_old)>0.001);
y1=y1n;
printf("      %f        %f/n",x1,y1);
x0=x1;
y0=y1;
}
printf("---------------------------/n");
printf("Hence y(%f)=%f",x1,y1);
}
float f(float x,float y)
{
float val;
val=y-x;
return val;
}
```

4.1.5.3 Write a program in C to solve a differential equation by Runge-Kutta Fourth Order method and take the following example. Given $\frac{dy}{dx} = y - x$, where $y(0) = 2$, find $y(0.2)$.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

void main ()
{
float h,x,x1,y1,x0,y0;
float k1,k2,k3,k4;
float f(float, float);
clrscr();
printf("Enter the step length h: ");
scanf("%f",&h);
printf("Enter the initial x and y values: ");
scanf("%f%f",&x0,&y0);
printf("Enter the x-value at which y can be dtermined: ");
scanf("f",&x);
printf("-----------------------/n");
printf("     x          y/n");
printf("-----------------------/n");
while(x0<x)
{
x1=x0+h;
k1=h*f(x0,y0);
k2=h*f(x0+h/2.0,y0+k1/2.0);
k3=h*f(x0+h/2,0,y0+k2/2.0);
k4=h*f(x0+h0+k3);
y1=y0+(k1+2*k2+2*k3+k4)/6.0;
printf("      %f      %f/n",x1,y1);
x0=x1;
y0=y1;
}
printf("-----------------------/n");
printf("Hence y(%f)=%f",x1,y1);
}
float f(float x, float y)
{
```

```
float val;
val=(x*x+y*y)/10.0;
return val;
```

4.1.5.4 Write a program in C to solve a differential equation by Milue's Predictor Corrector method and take the following example. Given $\frac{dy}{dx}=\frac{1}{10}(x^2+y^2)$, where $y(0)=2$, find $y(0.6)$, taking $h=0.1$.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

void main ()
{
float h,xf,x[4],y[4],x0,y0;
float x4,y4p,y4c,y4pold;
float k1,k2,k3,k4;
int i=1;
float f(float,float);
clrscr();
printf("Enter the step length h: ");
scanf("%f",&h);
printf("Enter the initial x and y values: ");
scanf("f%f",&x[0],&y[0]);
printf("Enter the x-value at which y can be dtermined: ");
scanf("%f",&xf);
printf("--------------------------------/n");
printf("        x               y/n");
printf("--------------------------------/n");
x0=x[0];
y0=y[0];
while(i<4)
{
x[i]=x0+h;
k1=h*f(x0,y0);
k2=h*f(x-+h/2.0,y0+k1/2.0);
k3=h*f(x0+h2.0,y0+k2/2.0);
k4=h*f(x0,y0+k3);
```

```
y[i]=y0+(k1+2*+2*k3+k4)6.0;
x0=x[i];
y0=y[i];
i=i+1;
}
while(x[3]<xf)
{
x4=x[3]+h;
y4p=y[0]+4.0*h/3.0*(2.0*f(x[1],y[1],y[1])-f(x[2],y[2])+2.0*f(x[3],y[3]));
do{
y4c=y[2]+h/3.0*(f(x[2],y[2])+4.0*f(x[3],y[3])+f(x4,y4p));
y4pold=y4p;
4yp=y4c;
}while(fabs)y4p-y4pold)>0.001);
printf("      %f       %f/n",x4,y4c);
for(i=0,i<3,i++)
{
x[i]=x[i+1];
y[i]=y[i+1];
}
x[3]=x4;
y[3]=y4c;
}
printf("----------------------/n");
printf("Hence y(%f)=%f",x4,y4c);
}
float f(float x,float y)
{
float val;
val=(x*x+y*y)/10.0;
return val;
}
```

4.1.6 EIGEN VALUE OF A MATRIX

4.1.6.1 Write a program in C to find the largest Eigen value and the corresponding Eigen vector by Power method. Test the program using the following matrix:

```
#include<stdio.h>
#include<conio.h>
```

```
#include<math.h>
#include<stdlib.h>

#define SIZE 5
#define ACCURACY 0.00001

void main()
{
float a[SIZE][SIZE],x[SIZE],y[SIZE],eigenvalue;
int n,i,j,k,iteration,id,ite_count=0;
clrscr();
printf("Enter the order of the square matrix: ");
scanf("%d",&n);
printf("Enter the elements of the matrix row-wise:\n");
for(i=0;i<n;i++)
for(j=0;j<n;j++)
scanf("%f",&a[i][j]);

iteration=0;
x[0]=1.0;
for(i=1;i<n,i++)
x[i]=0.0;

do{
iteration=iteration+1;
for(i=0;i<n;i++)
{
y[i]=0.0;
for(j=0;j<n;j++)
y[i]=y[i]+a[i][j]+a[i][j]*x[j];
}
eigenvalue=y[0];
for(i=0;i<n;i++)
y[i]=y[i]/eigenvalue;
id=0;
for(k=0;k<n;k++)
if(fabs(x[k]-y[k])>ACCURACY)
id=1;
if(id==1)
for(i=0;i<n;i++)
x[i]=y[i];
```

```
else
break;
ite_count++;
}while((1)&&(ite_count<=1000));
if(ite_count>100)
{
printf("By this method Eigen value can not be obtained");
exit(1);
}
printf("The required eigen value is ");
printf("%f/n",eigenvalue);
printf("The eigen vector is/n");
for(i=0;i<n;i++)
printf("y[%d]=%f/n",i+1,y[i]);
printf("is obtained at the iteration %d/n",iteration);
}
```

4.2 Statistical Problems

4.2.1 PREPARATION OF FREQUENCY TABLE AND HISTOGRAM

4.2.1.1 Write a program in C to form a frequency table for a distribution. Find it for following data :
43, 65, 51, 79, 25, 36, 49, 63, 49, 37, 40, 49, 75, 58, 65, 56, 76, 67, 54, 63, 73, 51, 49.

```
#include<stdio.h>
#include<conio.h>

#define XSIZE 50
#define GROUPNO 10

void main()
{
float x [XSIZE],max,min;
int i,j,low,high,n,f[XSIZE],NOC,length;
clrscr();
printf("Enter No. of Data: ");
scanf("%d",&n);
printf("Enter all data:/n");
for(i=0;i<n;i++)
scanf("%f",&x[i]);
```

```
min=x[0];
max=x[0];
for(i=1;i<n;i++)
{
if(min>x[i]
min=x[i];
if(max<x[i]
max=x[i];
}
printf("The Range is: %of/n",max-min);
printf("Enter class length: ");
scanf("%d",&length);
low=(int) min;
high=(int) max+1;
NOC=(high-low)/length+1;
for(i=1;i<=NOC;i++)
f[i-1]=0;
for(i=0;i<n;i++)
for(j=1;j<=NOC;j++)
if(low+(j-1)*length<=x[i]&& (x[i]<low+j*length))
        {
        f[j-1]=f[j-1]+1;
        break;
        }
printf("--------------------------/n");
printf("Class Interval/t/tFrequency/n");
printf("-------------------------/n");
for(j=1,j<=NOC;j++)
printf("%d---%d      \t%d/n",low+(j-1)*length,low+j*length,f[j-1]);
printf("--------------------------------/n");
}
```

4.2.1.2 Write a program in C to draw a Histogram for frequency distribution. Test the program using following data:

x	1	2	3	4
F	6	12	8	5

```
#include<stdio.h>
#include<conio.h>
```

```
#define SIZE 10

void main()
{
float x [SIZE];
int f[SIZE];
int i,j,k,n;
clrscr();
printf("Enter No. of Distinct Data: ");
scanf("%d",&n);
printf("Enter all data with its frequency:\n");
for(i=0;i<n;i++)
scanf("%f%d",&x[i],&f[i]);
printf("------------------------------/n");
printf("\t\t\tHISTOGRAM\n");
printf("\t|\n");
for(i=0,i<n;i++)
{
for(j=1,j<=3;j++)
{
if(j= =2)
printf("%0.2f  |",x[i]);
else
printf("\t|");
for(k=1,k<=f[i];k++)
printf("*");
if(j= =2)
printf("[%d]\n",f[i]);
else
printf("\n");
}
printf("\t|\n");
}
}
```

4.2.2 PROBLEMS ON SIMPLE FREQUENCY DISTRIBUTION

4.2.2.1 Write a program in C to find mean and standard deviation for discrete distribution. Test using following data :

X:	1	2	3	4	5	6	7	8	9
F:	8	10	11	16	20	25	15	9	6

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

#define SIZE 10

void main ()
{
float x[SIZE],mean=0.0,xf,std=0.0;
int n,i,f[SIZE],totalfre=0;
clrscr();
printif("Enter No. of Distinct Data: ");
scanf("%d",&n);
printf("Enter all data with its frequency:\n");
for(i=0;i<n;i++)
scanf("%f%d",&x[i],&f[i]);
printf("\tx\t\tf\tfx\n");
printif("------------------------------/n");
for(i=0;i<n;i++)
{
xf=x[i]*f[i];
mean=mean+xf;
totalfre=totalfre+f[i]
printf("\t%f\t%d\t%f\n",x[i],f[i],xf);
```

```
}
printf("--------------------\n");
printf("\tTotal\t\t%d\t%f\n",totalfre,mean);
mean=mean\totalfre;
for(i=0;i<n;i++)
std=std+f[i]*pow((x[i]-mean),2);
std=sqrt(std\totalfre);
printf("Therefore, mean=%f and standard deviation= %f\n",mean,std);
}
```

4.2.2.2 Write a program in C to find median and mode for discrete distribution. Test using following data:

X:	1	2	3	4	5	6	7	8	9
F:	8	10	11	16	20	25	15	9	6

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

#define SIZE 10

void main ()
{
float x[SIZE],median,mode;
int n,i,f[SIZE],cumufre[SIZE],totalfre,maxfre,maxindex;
clrscr();
printf("Enter No. of Distinct Data: ");
scanf("%d",&n);
prinft("Enter all data with its frequency:\n");
```

```
for(i=0;i<n;i++)
scanf("%f%d",&x[i];&f[i]);
printf("\tx\t\tf\tcumulative frequency\n");
printf("------------------------------\n");
cumufre[0]=f[0];
for(i=1,i<n;i++)
{
cumufre[i]=cumufre[i-1]+f[i];
printf("\t%f\t%d\t%d\n",x[i],f[i],cumufre[i]);
}
printf("---------------------------\n");
totalfre=cumufre[n-1];
pnntf("\tTotal\t\t%d\n",totalfre);
for(i=0;i<n;i++)
if(cumufre[i]>=totalfre/2)
{
printif("The required meadian is %f/n",x[i]);
break;
}
maxfre=f[0];
maxindex=0;
for(i=1;i<n;i++)
if(maxfre<f[i])
{
maxfre=f[i];
maxindex=i;
}
printif("The required mode is %f",x[maxindex]);
}
```

4.2.2.3 Write a program in C to find moments of any order for discrete distribution. Test using following data:

X:	1	2	3	4	5	6	7	8	9
F:	8	10	11	16	20	25	15	9	6

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

#define SIZE 10

float A;
int n;

void main ()
{
float x[SIZE];
int i,f[SIZE];
char choice;
float MEAN(float x[SIZE], int f[ISZE];
void MMNT(float x[SIZE], intf[SIZE];
clrscr();
printf("Enter No. of Distinct Data: ");
scanf("%d",&n);
```

```
printf("Enter all data with its frequency:\n");
for(i=0;i<n;i++)
scanf("f%d",&x[i],&f[i]);
printf("--------------------------\n");
printf("Enter the option which can be determined:\n");
printf("\t\t1.Moment about mean\n");
printf("\t\t2.Moment about other point\n");
printf("\tChoice:");
choice=getche();
switch(choice)
{
case '1':A=MEAN(x,f);
        MMNT(x,f);
        break;
case '2': printf('\n\tEnter the point:");
        scanf("%f",&A);
        MMNT(x,f);
        break;

default: printf("Please enter correct choice");
}
}
void MMNT(float x[SIZE], int f[SIZE]
{
FLOAT MMENT[SIZE];
int i,j,totalfre=0,MN;
printf("\n\tHow many moments are required?");
scanf("%d",&MN);
```

```
for(j=0;j<MN;j++)
mment[j]=0;
for(i=0;i<n;i++)
{
totalfre=totalfre+f[i];
for(j=0;j<MN;j++)
mment[j]=mment[j]+f[i]*pow(x[i]-A,j+1);
}
for(j=0;j<MN;j++)
mment[j]=mment[j]\totalfre;
printf("----------------------------------\n");
printf("\tTotal Frequency: %d\n",totalfre);
for(j=0,j<MN;j++)
printf("\t%d-Moment about the point %f: %f\n",j+1,A,mment[j]);
printf("----------------------------------\n");
getche();
}

float MEAN(float x [SIZE], int f[SIZE])
{
float mean=0;
int total=0,i;
for(i=0,i<n;i++)
{
total=total+f[i];
mean=mean+x[i]*f[i];
}
mean=mean/total;
```

```
    return mean;
}
```

5.0 Unit Summary

In this module, the methods to get solution of algebraic and transcendental equations, solution of a system of linear equations, to get the value of an Integration, to find interpolation, numerical solution of ordinary differential equation, Eigen value problem and some statistical methods are implemented in C.

6.0 Self Assessment Questions

1. Write a program in C to find the root of a polynomial equation.
2. Write a a program in C to find the root of a partial differential equation by finite difference method.
3. Write a program in C to find the solution of tri-diagonal equations.
4. Write a program in C to find the functioal value by Spline interpolation.
5. Write a program in C to find and replace a given letter in a given string.
6. Write a program in C to convert a name into abbreviated form.
7. Write a program in C to find all combinations of letters of a word.

7.0 Suggested further Readings

1. Balgurusamy E., Programming in ANSI C, Tata McGraw-Hill, 1992.
2. Venugopal K.R;, Programming with C, Tata McGraw-Hill, 2001.
3. Kumar R. and Agarwal R., Programming in ANSI C, Tata McGraw-Hill, 1993.
4. Xavier C., Fortran 77 and Numerical Methods, New Age International, 2000.

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-I

Paper-V — Group-B

Module No. - 59
PROGRAMMING IN C-III

Content :

MODULE 59 : Programming in C

1.0 Introduction

In less than a decade of its introduction, C has become a language of choice for software professionals. Some statistical methods, sorting techniques and some general problem, which are commonly used, are implemented in C language.

2.0 Objectives

In this module the following topics are implemented:

* Frequency distribution
* Sorting and Searching techniques
* String manipulation
* Miscellaneous problem.

3.0 Key Words and Study guides

Group frequency, Linear Search, Binary Search, Bubble sorting, Merge Sorting, Insertion Sorting, String concatenation

4.0 Main Discussion

4.1 Statistical Problems

4.1.1. PROBLEMS ON GROUP FREQUENCY DISTRIBUTION

4.1.1.1 Write a program in C to calculate the mean for the group frequency distribution. Test using following data:

Class Interval:	0–8	8–16	16–24	24–32	32–40	40–48
Frequency:	8	7	16	24	15	7

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

#define SIZE 10

void main ()
{
floatClassL[SIZE], ClassU[SIZE],mean=0.0;
float h,A,fd,xmid,d;
int n,i,f[SIZE],totalfre=0,max,AI;
```

```
clrscr();
printf("Enter No. of Distinct Data: ");
scanf("%d",&n);
printf("Enter all data with its frequency:\n");
printf("\nClass Interval\t\tFrequency\n");
for(i=0;i<n;i++)
scanf("%f%d",&ClassL[i]&ClassU[i],&f[i]);
h=ClassU[0]-ClassL[0];
max=f[0];
AI=0;
for(i=0;i<n;i++)
if(max<f[i]
{
max=f[i];
AI=i;
}
A(ClassU[AI]+ClassL[AI])/2;
printf("\nCL--CU\t\tx\tf\td\tfd\n");
printf("---------------------------------\n");
for(i=0;i<n;i++)
{
printf("%5.2f--%5.2f\t",ClassL[i],ClassU[i]);
xmid=(ClassU[i]+ClassL[i]\2;
printf("%5.2f\t%3d\t",xmid,f[i]);
d=(xmid-A)\h;
fd=f[i]*d;
printf("%5.2f\t%5.2f\n",d,fd);
mean=mean+fd;
totalfre=totalfre+f[i],
}
printf("--------------------------------\n");
printf("Total\t\t%3d\t\t%5.2f\n",totalfre,mean);
mean=A+h*mean\totalfre;

printf("Therefore, mean+%f\n",mean);
getche();
}
```

4.1.1.2 Write a program in C to calculate the median and mode for the group frequency distribution. Test using following data:

Class Interval:	0–8	8–16	16–24	24–32	32–40	40–48
Frequency:	8	7	16	24	15	7

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

#define SIZE 10

void main ()
{
float ClassL[SIZE];ClassU[SIZE],median, mode, h;
int n,i,f[SIZE],totalfre=0,median+ind,mode_id,maxfre,cumufre[SIZE];
clrscr();
printf("Enter No. of Distinct Data: ");
scanf("%d",&n);
printf("Enter all data with its frequency:\n");
printf('\nClass Interval\t\tFrequency\n");
for(i=0,i<n;i++)
scanf("%f%f%d",&ClassL[i},&ClassU[i],&f[i]);

printf("\nCL--CU\tf\tcf\n");
printf("----------------------------------\n");

cumufre[0]=f[0];
printf("5.2f--%5.2f\t",ClassL[0],ClassU[0];
printf[%3d\t%3d\t\n",f[0],cumufre[0];
for(i=1,i<n;i++)
{
printf("5.2f--%5.2f\t",ClassL[i],ClassU[i]);
cumufre[i]=cumufre[i-1]+f[i];
printf("%3d\t%3d\t\n",f[i],cumufre[i]);
}
totalfre=cumufre[n-1];
for(i=0,i<n;i++)
if(cumufre[i]<totalfre/2.0)
continue;
else
break;
median_ind=i;
h=ClassU[median_ind]-Class:[median_ind];
```

```
median=ClassL[median_ind]+h\f[median_ind]*(totalfre/2-comufre[median_ind_1]);
printf("-----------------------------\n");
maxfre=f[0];
for(i=0,i<n;i++)
if(maxfre<f[i])
{
maxfre=f[i];
mode_id=i;
}
h=CassU[mode_id]-ClassL[mode_id];
mode=ClassL[mode_id]+(h*[mode_id]-f[mode_id-1]))/(2*f[mode_id]-f[mode_id-1]-f[mode_id+1]);
printf("Therefore,\tmedian=%f and mode=%f\n",median,mode);
getche();
}
```

4.1.2 ON BIVARIATE DISTRIBUTION

4.1.2.1 Write a program in C to find out a correlation coefficient for a set of points (x_i, y_i), i = 1, 2,....n and find it using following data :

X	65	66	67	67	68	69	70	72
Y	67	68	65	68	72	72	69	71

```
#include<stdio.h>
#include<conio.h>
#include<math.h>

#define NP 10

void main ()
{
float x[NP],y[NP],r;
float xysum=0,xsum=0,ysum=0,x2sum=0,y2sum,xbar,ybar;
int n,i;
clrscr();
printif("Give No. of points (x,y); ");
scanf("%d",&n);
printf("Input all x and values of all points: \n");
for(i=0;i<n;i++)
scanf("%f%f",&x[i],&y[i]);
for(i=0,i<n;i++)
{
```

```
xysum+=x[i]*y[i];
xsum+=x[i];
ysum+=y[i];
x2sum+=x[i]*x[i];
y2sum|=y[i]*y[i];
}
xbar=xsum/n;
ybar=ysum/n;
r=(xysum/n-xbar*ybar)/sqrt((xqrt((x2sum/n-xbar*xbar)*(y2sum/n-ybar*ybar));
printf("So The required correlation coefficient=%f",r);
}
```

4.1.2.3 Write a program in C for fitting a straight line through a set of points (x_i, y_i), i = 1, 2,....n. Using following set of points, test it.

x	4	6	8	10	12
y	13.72	12.90	12.01	11.14	10.31

```
#include<stdio.h>
#include<conio.h>

#define NP 10

void main ()
{
float x[NP],y[NP],m,c;
float sysum=0,xsum=0,ysum=0,x2sum=0;
int n,i;
clrscr();
printf("Give No. of points (x,y):");
scanf("%d",&n);
printf("Input all x and values of all points:\n");
for(i=0;i<n;i++)
scanf("%f%f",&x[i],&y[i]);
for(i=0,i<n;i++)
{
xysum+=x[i]*y[i];
xsum+=x[i];
ysum+=y[i];
x2sum+=x[i]*x[i];
}
```

```
m=(n*xysum-xsum*ysum)/(n*x2sum-xsum*xsum);
c=(ysum-m*xsum)/n;
printf("So The required equation of the line is ");
printf("y = %fx + %f"m,c);
}
```

4.2 Searching and Sorting

4.2.1 SEARCHING

4.2.1.1 Write a program in C to search a number from a list of numbers by Linear Search Technique.

```
#include<stdio.h>
#include<conio.h>

#define MAX 10

void main ()
{
float num[MAX],item;
int i,n,id;
clrscr();
printf("How many numbers are in the list?")'
scanf("%d",&n);
printf("Enter all elements in the list:\n");
for(i=0;i<n;i++)
scanf("%f",&num[i]);
printf("Please enter the searching number:");
scanf("%f",&item);
id=0;
for(i=0;i<n;i++)
if(num[i]= = item)
{
printf("The searching number %f is %d-th position in the list\n",item,i+1);
id=1;
if(id= =)
printf("The searching number %f is not in the list", item);
getch();
}
```

4.2.1.2 Write a program in C to search a number from a sorted list of numbers by Bonary Search Technique.

```
#include<stdio.h>
#include<conio.h>

#define MAX 10

void main ()
{
float num[MAX],item;
int i,n,id,lindex,uindex,index,count=0;
clrscr();
printf("How many numbers are in the list?");
scanf("%d",&n);
printf("Enter all elements in the list\n(All elements must be sorted):\n");
for(i=0;i<n;i++)
scanf("%f",&num[i]);
printf("Please enter the searching number: ");
scanf("%f",&item);
id=0;
lindex=0;
uindex=n-1;

do{
mindex=(lindex+uindex)/2;
count=count+1;
if(num[mindex]= = item)
{
printf("The searching number %f is %d-th position in the list\n",item,mindex+1);
id=1;
break;
}
else if(num[mindex]>item)
uindex=mindex-1;
else
lindex=mindex+1;
}while(count<=n);
if(id= =)
print("The searching number %f is not in the list",item);
getch();
}
```

4.2.2 SORTING

4.2.2.1 Write a program in C to arrange a list of numbers in ascending orders by Bubble sort technique.

```
#include<stdio.h>
#include<conio.h>

#define MAXSIZE 10

void main ()
{
float num[MAXSIZE],temp;
int i,j,NoElement;
clrscr();
printf("Enter Number of Element in the List: ");
scanf("%d",&NoElement);
printf("Enter List Elements;\n");
for(i=0,i<NoElement;i++)
scanf("%f",&num[i]);
for(i=0;i<NoElement-1,i++)
for(j=0;j<NoElement-1-i,j++)
if(num[j]>num[j+1])
{
temp=num[j];
num[j]=num[j+1];
num[j+1]=temp;
}
printf("\nThe Sorted List by Bubble Sort Technique is\n");
for(i=0;i<NoElement;i++)
printf("%f\n",num[i]);
getche();
}
```

4.2.2.2 Write a program in C to arrange a list of numbers in ascending orders by Insertion sort technique.

```
#include<stdio.h>
#include<conio.h>

#define MAXSIZE 10
```

```
void main ()
{
float num[MAXSIZE];temp;
int i,j.NoElement;
clrscr();
printf("Enter Number of Element in the List: ");
scanf("%d",&NoElement);
printf("Enter List Elements:\n ");
for(i=0;i<NoElement;i++)
scanf("%f",&num[i]);
for(i=1;i<NoElement;i++)
{
j=i-1;
temp=num[i];
while(temp<num[j])
{
num[j+1]=num[j];
j=j-1;
}
num[j+1]=temp;
}
printf("n\The Sorted List by Insertion Sort Technique is\n ");
for(i=0;i<NoElement;i++)
printf("%f\n",num[i]);
getche();
}
```

4.2.2.3 Write a program in C to sort some numbers in ascending order by Selection sort method.

```
#include<stdio.h>
#include<conio.h>

#define   MAXSIZE    10

void main ()
{
float num[MAXSIZE],temp;
int i,j.NoElement,min;
clrscr();
printf("Enter Number of Element in the List: ");
```

```
scanf("%d",&NoElement);
printf("Enter List Elements:\n");
for(i=0;i<NoElement;i++)
scanf("%f",&num[i]);
for(i=0,i<NoElement-1;i++)
{
min=i;
for(j=j+1;j<NoElement;j++)
if(num[min]>num[j])
min=j;
if(min!=i)
{
temp=num[i];
num[i]=num[min];
num[min]=temp;
}
}
print("\nThe Sorted List by Selection Sort Technique is\n");
for(i=0;i<NoElement;i++)
printf("f\n",num[i]);
getche();
}
```

4.2.2.4 Write a program in C to sort some numbers in ascending order by Merge Sort method. Test using 12, 45, 123, 22, 89, 16, 543, 66, 120, 80, 10, 30 and 150.

```
#include<stdio.h>
#include<conio.h>

#define    MAXSIZE    15

int array[MAXSIZE],final[MAXSIZE];
void Merge();
int Read_mer();
void Display();
int Read_mer()
{
int n,m,i;
printf("Enter how many elements do you want in the array: ");
scanf("%d",&n);
for(i=0;i<n;i++)
```

```
{
printf("\nEnter%d-th element in the arry: ",i+1);
scanf("%d",&array[i]);
}
return n;
}
void Display(int t)
{
int i;
printf("\n  ");
printf("THE RESULT OF THE MERGE SORT TECHNIQUE\n");
printf("\n\nThe Sorted List is: ");
for(i=0;i<t;i++)
printf("%d ",array[i]);
}
void Merge(int low,int mid,int high)
{
int h,j,k,i;
h=low;
j=mid+1;
i=low;

while((h<=mid)&&(j<=high))
{
if(array[h]<=array[j])
{
final[i]=array[h];
h=h+1;
}
else
{
final[i]=array[j];
j=j+1;
}
i=i+1;
}
if(h>mid)
for(k=j;k<=high;k++)
{
final[i]=array[k];
i=i+1;
```

```
}
else
for(k=h;k<=mid;k++)
{
final[i]=array[k];
i=i+1;
}
for(k=low;k<=high;k++)
array[k]=final[k];
}

void MergeSort(int low,int high)
{
int mid;
if(low<high)
{
mid=(int)((low+high)/2);
MergeSort(low,mid);
MergeSort(mid+1,high);
Merge(low,mid,high);
}
}

void main()
{
int low=0,high;
clrscr();
high=Read_mer();
MergeSort(low,high-1);
Display(high);
getch();
}
```

4.2.2.5 Write a program in C to sort some numbers in ascending order by Quick sort method. Test using 12, 45, 123, 22, 89, 16, 543, 66, 120, 80, 10, 30 and 150.

```
#include<stdio.h>
#include<conio.h>

#define   MAXSIZE   20
```

```
void Exchange(int A[max],int f,int 1)
{
int temp;
temp=A[f];
A[f]=A[l];
A[l]=temp;
}

int Partition (int A[max],int p, int r)
{
int x,i,j,loc;
x=A[p];
i=p;
j=r;
for(;;)
{
while((A[j]>=x)&&(j>p))
j--;
while((A[j]<x)&&(i<r))
i++;
if(i,j)
Excahnge(A,i,j);
else
{
loc=j;
break;
}
}
return loc;
}

void QuickSort(int A[max],int p,int r)
{
int q;
if(p<r)
{
q=Partition(A,p,r);
QuickSort(A,p,q);
QuickSort(A,q+1,r);
}
}
```

```
void main()
{
int x[max],i,n,j=0;
//void QuickSort(int A[max],int p, int r);
clrscr();
printf("\nEnter how many number you want in the list: ");
scanf("%d",&n);
for(i=0,i<n;i++)
{
printf("\nEnter the %d-th element: ", i+1);
scanf("%d",&x[i]);
}
QuickSort(x,j,n);
printf("\n\nTHE SORTED LIST IS;");
for(i=0;i<n;i++)
printf("%d\t",x[i]);
printf("\n\nPress any key to continue");
getch();
}
```

4.2.2.6 Write a program in C to sort some numbers in ascending order by Heap sort method. Test using 12, 45, 123, 22, 89, 16, 543, 66, 120, 80, 10, 30 and 150.

```
#include<stdio.h>
#include<conio.h>

#define    max    20

int heap[max+1],size;

int heap_size(int A[max])
{
inti=0;
while(A[i]!=-99)
i++;
return i;
}
int let_child(int x)
{
```

```
return 2*x+1;
}

int right_child(int x)
{
return 2*x+2;
}

void exchange(int*T,int, s, int L)
{
int temp;
temp=*(T+s);
*(T+s)=*(T+L);
*(T+L)=temp;
}

void heapify(int A[max],int n)
{
int l,r,largest;
l=left child(n);
r=right_child(n);
if((l<=size)&&(A[1]>A[n]))
largest=1;
else
largest=n;
if((r<=size)&&(A[r]>A[largest]))
largest=r;
if(largest~=n)
{
exchange(A,n,largest);
heapify(A,largest);
}
}
void build_heap(int A[max])
{
int i,n;
size=heap_size(A);
for(i=size/2;i>=0,i--)
heapify(A,i);
}
```

```
void HeapSort(int A[max])
{
int length,i;
build_heap(A);
length=heap_size(A);
size=size-1;
for(i=length-1;i>=1,i--)
{
exchange(A,0,i);
size--;
heapify(A,0);
}
}

void main()
{
int n,i;
clrscr();
printf("\nEnter how many element do you want in the heap: ");
scanf("%d",&n);
for(i=0;i<n;i++)
{
printf("\nEnter%d-th element; ", i+1);
scan("%d",&heap[i]);
}
heap[n]=-99;
build_heap(heap);
print("\n");
printf("     ");
printf("After maintaining the heap property\n");
printf("       ");
printf("--------------------\n\n");
for(i=0;i<n;i++)
{
printf("          ");
printf("The %d-th element:",i+1);
printf("%d\n\n",heap[i]);
}
printf("Press any key to sort the element by heapd sort...\n\n");
getch();
```

```
HeapSort(heap);
printf("The sorted array is:");
for(i=0;i<n;i++)
printf("%d\t",heap[i]);
printf("\nPress any key to terminate....");
getch();
}
```

4.3. String Manipulation

4.3.1 Write a program in C to find a number of occurrences of a letter in a given string.

```
#include<stdio.h>
#include<conio.h>
#include<stdlib.h>

#define   SIZE   50

void main()
{
char STR[SIZE],SpLetter,cchar;
int i,count,NoLetter=0;
clrscr();
printf("When completed, press 'RETURN' key.\n");
printf("Give the String;\n");
count=0;
while((cchar=getchar())!'\n')
STR[count++]=cchar;
STR[count]='\0';

printf("Enter Searching Letter: ");
SpLetter=getchar();
```

```
if(STR[0]= ='\0')
{
printf("No string");
exit(1);
}
else
{
for(i=0,STR[i]!='\0',i++)
if(STR[i]= =SpLetter)
        ++NoLetter;
}
printf("\n");
printf("No. of %c in the given string is --> %d\n",SpLetter,NoLetter);
getche();
}
```

4.3.2 Write a program in C to check whether a string is palindrome or not.

```
#include<stdio.h>
#include<conio.h>
#include<string.h>

#define    SZ    25

void main()
{
char string[SZ],cr;
```

```
int i,c=0,len,id;
clrscr();
printf("Enter the a string: ");
while((cr=gethar()!='\n')
string[c++]=cr;
string[c]='\0';
len=strlen(string);
id=0;
for(i=0;i<len/2,i++)
if(string[i]!=string[len-i-1])
{
id=1;
break;
}
if(id==)
printf("%s is Palindrome",string);
else
printf("%s is not palindrome",string);
getche();
}
```

4.3.3 Write a program in C to rewrite a name wtih surname first, allowed by a comma and the initials of the first middle names. Test taking following names:

Bimal Krishna Bharadra, Amal Chandra Samanta, Shyam Charan Manna.

```
#include<stdio.h>
#include<conio.h>
#include<string.h>
```

```
#define   TOTAL   10

void main()
{
int i,j,n;
char fname[TOTAL][15],sname[TOTAL][15],surname[TOTAL][15];
char name[TOTAL][20],TEMP[5];
clrscr();
printf("Enter No. of Names: ");
scanf("%d",&n);
printf("Enter all Names:\n");
for(i=0;i<n;i++)
scanf("%s%s%s",fname[i],sname[i],surname[i]);
for(i=0;i<n;i++)
{
strcpy(name[i];surname[i]);
strcat(name[i],",");
temp[o]=fname[i][0];
temp[1]='\0';
strcat(name[i];temp);
strcat(name[i], ".");
temp[0]=sname[i][0];
temp[1]='\0';
strcat(name[i],temp);
}
printf("List of Name with surname first\n");
for(i=0;i<n;i++)
```

```
printf("/t%-20s\n",name[i]);
}
```

4.3.4 Write a program in C to reverse a string.

```
#include<stdio.h>
#include<conio.h>
#include<string.h>

#define    SIZE    25

void main()
{
char word[SIZE],reve[SIZE];
int i,j,len;
clrscr();
printf("Enter a word: ");
gets (word);
len=strlen(word);
j=len;
reve[j--]='\0';
i=0;
while(word[i]!='\0')
reve[j--]=word[i++];
printf("\n The original string: %s\n",word);
printf("The reverseed string: %s\n",reve);
getche();
}
```

4.3.5 Write a program in C to sort a list of names in alphabetic order,

```
#include<stdio.h>
#include<conio.h>
#include<string.h>

#define   SZ   15
#define   SL   15

void main()
{
char name[SZ][SL],temp[SL];
int i,j,n;
clrscr();
printf("Enter No. of Names: ");
scanf("%d",&n);
printf("Gvie all names:\n");
for(i=0;i<n;i++)
scanf("%s",name[i]);
printf("\n********Unsorted list ********\n");
for(i=0;i<n;i++)
printf("%s\n",name[i]);
for(i=0;i<n-1,i++)
for(j=0;j<n-i;j++)
if(strcmp(name[j],name[j+1]>0)
{
strcpy(temp,name[j]);
```

```
strcpy(name[j],name[j+1]);
strcpy(name[j+1],temp);
}
printf("***********SORTED LIST***********\n");
for(i=0;i<n;i++)
printf("%s\n",name[i]);
getche();
}
```

4.3.6 Write a program in C to search a given word in a given string.

```
#include<stdio.h>
#include<conio.h>
#include<string.h>

#define    STR_SE    50
#define    WRD_SE    10

void main()
{
char str[STR_SE],temp[WRD_SE],term[WRD_SE];
char c;
int i,j,k,found;
clrscr();
printf("Enter the String: \n");
gets(str);
printf("Enter the Searching Term:\n");
gets(term);
```

```
printf("\n");
i=0;
j=0;
found=0;
while(str[i]!='\0')
{
c=str[i];
if(c!=' ')
{
temp[j]=c;
}
if(c = = ' ')
{
temp[j]='\0';
if(strcimp(temp,term)= = 0)
{
found=1;
printf("%s is in the string\n",term);
}
j=0;
}
else
j++;
i++;
}
if(found= =0)
printf("%s is not found",term);
}
```

4.3.7 Write a program in C to count the characters, words and line in a text.

```
#include<stdio.h>
#include<conio.h>
#include<string.h>

#define SIZE 50

void main()
{
char TEXT[SIZE],cchar;
int i, count,end=0,NoChar=0,No Word=0,NoLine=0;
clrscr();
printf("Give one space after each word.\n");
printf("When completed, press 'RETURN' key.\n");
while(end= =0)
{
count=0;
while((cchar=getchar())!='\n')
TEXT[count++]=cchar;
TEXT[count]='\0';
if(TEXT[0]= ='\0')
break;
else
{
NoWord++;
for(i=0,TEXT[i]!='\0';i++)
if(TEXT[i]= =' '|| TEXT[i]= ='\t')
```

```
        NoWord++;
    }
    NoLine=NoLine+1;
    NoChar=NoChar+strlen(TEXT);
    }
    printf("\n");
    printf("No. of Lines=%d\n",NoLine);
    printf("No. of Words = %d\n", NoWord);
    printf("No. of Characters = %d\n",NoChar);
    }
```

4.5 Miscellaneous Problems

4.5.1 Write a program in C to generate some random numbers.

```
#include<stdio.h>
#include<conio.h>

#define   MULTIPLIER   25173
#define   INCREMENT    13849
#define   MODULUS      65536
#define   SEED         319347905

double random(unsigned int seed0)
{
static unsigned int number;
static int initialized=0;
if(!initialized)
{
```

```
number=seed0%MODULUS;
initialized=1;
}
number=(MULTIPLIER * number+INCREMENT)%MODULUS;
return (double)number/(MODULUS-1);
}
void main()
{
int i,num,seed;
clrscr();
printf("How many random numbers are required? ");
scanf("%d",&num);
printf("Enter Seed Value: ");
scanf("%d"),&seed);
for(i=1,i<=num,i++)
printf("%f",random(seed));
getch();
}
```

Note : The first time *random* is called; the variable *initialized* is set to 0. Therefore, *number* is initialized to *see0*, and *initialized* is set to 1. Since both *number* and *initialized* are static variables, their values are preserved across function calls, and the variable *initialized* is initialized only once. Consequently, *number* is also initialized with *seed0* only once, and every call to *random* generates a new psedo-random value using the previous value saved in *number.*

4.5.2 Write a program in C to generate all prime numbers between two specified numbers.

```
#include<stdio.h>
#include<ccnio.h>

void main()
{
int num,i,j,LB,UB,id;
clrscr();
printf("Give the Lower boundary: ");
scanf("%d",&LB);
printif("Give the upper boundary: ");
scanf("%d",&UB);
printf("\nThe followings are prime between %d and %d\n", LB,UB);
for(i=LB;i<=UB;i++)
{
id=0;
for(j=2;j<=i/2;j++)
if(= =(i/j)*j)
{
    id=1;
    break;
}
if(id= =0)
printf("%d\n",i);
}
getche();
}
```

4.5.3. Write a program in C to generate first some Fibonacci numbers.

```
#include<stdio.h>
#include<conio.h>

void main()
int FibNo,count,F1,F2,F;
clrscr();
printif("How many Fibonacci Numbers are required? ");
scanf("%d",&FibNo);
if(FibNo==1)
printf("%d",0);
elseif(FibNo<=2)
printf("%d\n%d",0,1);
else
{
printf("%d\n%d\n",0,1);
count=2;
F1=0;
F2=1;
while(count<FibNo)
{
count=count+1;
F=F1+F2;
printf("%d\n",F);
F1=F2;
F2=F;
}
{
getche();
}
```

4.5.4 Write a program to generate PASCAL's triangle.

```
#include<stdio.h>
#include<conio.h>
void main()
{
    int binom,p,q,r,x;
    clrscr();
```

```
bonom=1;
q=0;
printf("Input the number of rows: ");
scanf("%d",&p);
printf("Pascals triangle:\n");
while(q<p)
{
for(r=40-3*q;r>0;--r)
printf(" ");
for(x=0;x<=q,++x)
{
    if((x= =0) || (q= =0))
    bonom=1;
    else
    bonom=(bonom*(q-x+1)/x;
    printf("%6d",bonom);
"
printf("\n");
q++;
}
}
```

4.5.5 Write a program in C to prepare a multiple-choice test.

```
#include<stdio.h>
#include<conio.h>

#define  STUD   3
#define  QUES   25

void main()
{
char answer[QUES+1],response[QUES+1];
int count,i,student,n,correct[QUES+1];
print("Enter Answers to the Questionas\n");
for(i=0;i<QUES;i++)
scanf("%c",&answer[i]);
answer[i]='\0';
for(student=1;student<=STUD;student++)
{
count=0;
```

```
printf("\n");
printf("Input Responses of Student-%d\n",student);
for(i=0;i<QUES;i++)
scanf("%c",&response[i]);
response[i]='\0';
for(i=0;i<QUES;i++)
correct[i]=0;
for(i=0;i<QUES;i++)
if(response[i]= =answer[i])
{
        count=count+1;
        correct[i]=1;
}
printf("\n");
printf("Student-%d\n",student);
printf("Score is %d out of %d\n",count,QUES);
printf("Response to the Questions below are wrong\n");
n=0;
for(i=0;i<QUES;i++)
if(correct[i]= =0)
{
printf("%d",i+1);
n=n+1;
}
if(n= =0)
printf("NIL\n");
printf("\n");
}
}
```

Note: In the above program the following array are used:

answer[i]: To store correct answers of the questions
response[i]: To store response of students
correct[i]: To identify Questions that are answered correctly.

4.5.6 Write a program in C to check a number for a palindrome.

```
#include<stdio.h>
void main()
{
int n,num,digit,sum=0,rev=0;
```

```
printf("Input the number: ");
scanf("%d",&num);
n=num;
do
{
        digit=num%10;
        sum+=digit;
        rev=rev*10+digit;
        num/=10;
        }while(num!=0);
        printf("sum of the digits of the number=%4d\n",sum);
        printf("Reverse of the number=%7d\n",rev);
        if(n==rev)
                printf("The number is a palindrome\n");
        else
                printf("The number is not a palindrome\n");
}
e.g.: 1221
```

4.5.7 Write a program to convert the upper case to lower case and vice-versa.

```
#include<stdio.h>
#include<ctype.h>
void upper(char instr[])
{
        inti=0;
        while(instr[i])
        {
        instr[i]=toupper(instr[i]);
        i++;
}
}

void lower(char instr[])
{
inti=0;
while(instr[i])
{
instr[i]=tolower(instr[i]);
i++;
}
```

```
}

void main()
{
        char instr[80],inchar;
        printf("Input the string: ");
        gets(instr);
        printf("To upper or lower case(u/l)?");
        scanf("%c",&inchar);
        switch(inchar)
        {
        case 'u':
        case 'U':upper(instr);
                break;
        case 'l':
        case 'L': lowerInstr);
                break;
        default:printf("Invalid input!\n");
        }
        printf("The converted string is...\n");
        printf("%s\n",instr);
```

4.5.8 Write a program to find the real roots of a quadratic equation.

The roots of the quadratic equation $ax^2+bx+c=0$, where, a, b and c are real and a is not equal to zero, are given by

$$\left(-b \pm \sqrt{b^2-4ac}\right)/2a$$

The roots are equal when the discriminant $b^2-4ac \geq 0$. If $b^2-4ac=0$, the roots are real and equal.

```
#include<stdio.h>
#include<math.h>
int main()
{
        float a,b,c;
        float dis,root,root1,root2;
        scanf("%f%f%f",&a,&b,&c);
        if(a= =0)
        {
        printf("Not a quadratic equation\n");
        return 1;
        }
        dis=b*b-4*a*c;
```

```
if(dis<0)
print("No real roots\n");
else if(dis==0)
{
root=-b/(2*a);
printf("Two identical roots: %f\n",root);
}
else
{
root1=(-b+sqrt(dis))/(2*a);
root2=(-b-sqrt(dis))/(2*a);
printf("Two distinct roots: %of%f\n",root1,root2):
}
return 0;
}
```

4.5.9 Write a program in C to product two matrices.

```
#include<stdio.h>
#include<conio.h>
#define row 4
#define col 4
void main()
{
float a[row][col],b[row][col],c[row],[col];
int ra, ca, rb, cb;
int i,j,k;
clrscr();
printf("Enter the no. of orders of A-Matrix: ");
scanf("%d%d,&ra,&ca);
printf("Enter the no. of orders of B-Matrix: ");
scanf("%d"d",&rb,&cb);
printf("Enter all elements of A-matrix row-wise:\n");
for(i=0;i<ra;i++)
for(j=0;j<ca;j++)
scanf("%f",&a[i][j]);
printf("\nEnter all elements of B-matrix row-wise:\n");
for(i=0;i<rb;i++)
for(j=0;j<cb;j++)
scanf("%f",&b[i][j];
if(ca==rb)
```

```
{
for(i=0;i<ra;i++)
for(j=0;j<cb;j++)
{
c[i][j]=0.0;
for(k=0;k<ca;k++)
c[i][j]=c[i][j]+a[i][k]*b[k][j];
}
printf("Product of two matrices:\n");
for(i=0;i<ra;i++)
{
for(j=0;j<ca;j++)
printf("%0.2f ",c[i][j]);
printf("\n");
}
}
else
printf("Matrix Multiplication is not compatible");
}
```

4.5.10 Write a program to count the number of vowels, consonents and space in a line.

```
#include<stdio.h>
#include<string.h>
#include<ctype.h>
#include<conio.h>
#define NTYPES  4
enum CHAR_TYPE {NONE,VOWEL,CONSONENT,SPACE};
char vowels[]="aeiouAEIOU";
enum CHAR_TYPE get_type(char c)
{
enum CHAR_TYPE retval=NONE;
if(c==' ')
retval=SPACE;
else if(isalpha(c))
{
        if(strchr(vowels,c))
        retval=VOWEL;
        else
        retval=CONSONENT;
        }
        return retval;
```

```
}

void main()
{
char input[50];
int len, i;
int counts[NTYPES]={0,0,0,0};
puts(Enter a line of a text:");
gets(input);
len=strlen(input);
for(i=0;i<len;i++)
    {
    int index=get_type(input[i]);
    counts[index]++;
    }
printf("Results:\n");
printf("Vowels:%d\n",counts[VOWEL]);
printf("Consonents:%d\n",counts[CONSONENT]);
prinf("Spaces: %d\n",counts[SPACE]);
}
```

4.5.11 Write a program in C to find the inverse of a matrix.

```
#include<stdio.h>
#include<conio.h>
#include<stdlib.h>

#define SIZE 10

void main()
{
float a[SIZE][SIZE],b[SIZE][SIZE],ratio;
int i,j,k,n;
clrscr();
printf("Enter the order of a square matrix: ");
scanf("%d",&n);
printf("Enter all elements of the matrix row-wise:\n");
for(i=0;i<n;i++)
for(j=0;j<n;j++)
scanf("%f",&a[i],[j]);
```

```
for (i=o;i<n;i++)
for(j=0;j<n;j++)
if(i==j)
b[i][j]=1.0;
else
b[i][j]=0.0;

for(k=0;k<n;k++)
for(i=0;j<n;i++)
if(i==k)
continue;
else
{
if(a[k][k]==0)
{
printf("METHOD FAILS");
exit(1);
}
ratio=a[i][k]/a[k][k];
for(j=0;j<n;j++)
{
a[i][j]=a[i][j]-ratio*a[k][j];
b[i][j]=b[i][j]-ratio*b[k][j];
}
}
for(i=0;i<n;i++)
for(j=0;j<n;j++)
b[i][j]=b[i][j]/a[i][i];
printf("\nInversion of the matris is:\n");
for(i=0;i<n;i++)
{
for(j=0;j<n;j++)
printf("f\t",b[i][j]);
printf("\n");
}
}
```

4.5.12 Write a program to find a product of two polynomials.

```
#include<stdio.h>
#include<conio.h>
```

```
#define  SZ   10
void main()
{
float a[SZ+1,b[SZ+1],c[2*SZ+1];
int i,j,k,m,n,order_c;
clrscr();
printf("Enter the Order of 1-st Polynomial");
scanf("%d",&m);
printf("Enter the Coefficients:\n");
for(i=0;i<=m,i++)
scanf("%f",&a[i]);
printf("Enter the Order of 2-nd Polynomial: ");
scanf("%d",&n);
printif("Enter the Coefficients:\n");
for(j=0;j<=n;j++)
scanf("%f",&b[j]);
order_c=m+n;
for(i=order_c;i>=o;i--)
{
c[i]=0.0;
for(j=0;j<=m;j++)
for(k=0;k<=n;k++)
if(j+k= =i)
c[i]=c[i]+a[j]*b[k];
}
printf("\n\nThe Product Polynomial is\n");
for(i=0;i<=order_c;i++)
if(i= =order_c)
printf("%0.2fx[%d]",c[i],order_c-i);
else
printf("%0.2fx[%d]+",x[i],order_c-i);
getche();
}
```

5.0 Unit Summary

In this unit, some statistical problems, sorting, searching techniques, string manipulation and some imscellaneous problems are implemented in C language.

6.0 Self assessment questions

1. Write a program in C to find the root of a polynomial equation.
2. Write a program in C to find the root of a partial differential equation by finite difference method.
3. Write a program in C to find the solution of tri-diagonal equations.
4. Write a program in C to find the functional value by Spline interpolation.
5. **Write a program in C to find and replace a given letter in a given string.**
6. **Write a program in C to convert a name into abbreviated form.**
7. **Write a program in C to find all combinations of letters of a word.**

7.0 Suggested further Readings

1. Balgurusamy E., Programming in ANSI C, Tata McGraw-Hill, 1992.
2. Venugopal K.R., Programming with C, Tata McGraw-Hill, 2001.
3. Kumar R. and Agrawal R., Programming in ANSI C, Tata McGraw-Hill, 1993.
4. Xavier C., Fortran 77 and Numerical Methods, New Age International, 2000.

VIDYASAGAR UNIVERSITY

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M.Sc. in Applied Mathematics with Oceanology and Computer Programming

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PROGRAMMING IN FORTRAN-77

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MODULE 60 : Programming in FORTRAN-77

1.0 Introduction

Numerical analysis is concerned with methods, which give numerical solution to mathematical problems by arithmetic operations on numbers. In Applied Mathematics, Theoretical Physics and Engineering, the ultimate problem is to compute numerical results using certain data. In modern times, Statistics is viewed not as a mere device for collecting numerical data but as a means of developing sound techniques for their handlign and analysis and drawing valid inferences from them. With the advent of modern high-speed digital computers the numerical analysis and statistics of today are two very different disciplines. To implement numerical and statistical methods FORTRAN-77 programming is very useful to day.

2.0 Objectives

In this module the following topics are implemented:

- Solution of algebraic and transcendental equations
- Solution of a system of linear equations
- Numerical Integration
- Interpolation
- Numerical solution of ordinary differential equation
- Eigen value problem
- Some Statistical methods
- Some Miscellaneous Problems

3.0 Key Words and Study guides

Interpolation, Integration, Linear equations, Algebraic equation, transcendental equation. Group frequency, Linear Search, Binary Search, Bubble sorting, Merge Sorting, Insertion Sorting, String concatenation

4.0 Main Discussion

4.1 Numerical Problems

4.1.1 SOLUTION OF ALGEBRAIC AND TRANCENDENTAL EQUATIONS

4.1.1.1 Write a programme to find a real root of an equation by Bisection method. Test the program taking the example $f(x) = x^3 - x - 1 = 0$.

```
C    BISECTION METHOD
     F (X) = X**3 - X - 1.0
     COUNT=0
     PRINT*, 'ENTER THE ACCURACY'
     READ*, EPS
70   PRINT*, 'ENTER THE LIMITS OF THE ROOT'
     COUNT=COUNT+1
     READ*,A,B
     FA=F(A)
     FB=F(B)
     IF(FA*FB.LT.0.0) THEN
     GOTO 60
     ELSE
     IF(COUNT.EQ.50)THEN
     PRINT*, 'NO ROOT IN THE GIVEN INTERVAL'
     STOP
     ENDIF
     PRINT*, 'NO ROOT IN THIS INTERVAL: (',A,B,')'
     GOTO 70
     ENDIF
60   X=(A+B)/2
     FX=F(X)
     IF (ABS(FX).LT.EPS) GOTO 50
     IF (FX*FA).LT.0) THEN
     B=X
     GOTO 60
     ELSEIF (FX*FB).LT.0) THEN
     A=X
```

```
      GOTO 60
      ENDIF
50    PRINT*, 'THE ROOT IS', X
      STOP
      END
```

4.1.1.2 Write a programme to find a real root of an equation by Iteration method. Test the program taking the example $2x = \cos x + 3$.

```
      F(X)=X*X-2*X+1
      PHI (X)=1.0/(2-X)
      PRINT*,'ENTER INITIAL VALUE:'
      READ,X0
      PRINT*, 'ENTER THE MAXIMUM NUMBER OF ITERATIONS:'
      READ*,N
      I=0
10    IF (2-X0.EQ.0) THEN
      PRINT*, 'METHOD FAIL'
      STOP
      ENDIF
      X1=PHI(X0)
      IF(ABS(X1-X0(.LT.1.0E-4) GOTO 20
      X0=X1
      I=I+1
      IF(I.LT.N) GOTO 10
      PRINT*, 'NOT CONVERGE'
      GOTO 30
20    PRINT*, 'ROOT IS', X1
30    STOP
      END
```

4.1.1.3 Write a program to find a real root of an equation by Newton-Raphson's method. Test the program taking the example $x \sin x + \cos x = 0$.

```
C     TO FIND THE ROOT OF AN EQUATION F(X)=0
```

```
      F(X)=X**3-5.0*X+3.0
      DF(X)=3.0*X*X-5.0
      COUNT=0
10    PRINT*, 'ENTER THE INTERVAL IN WHICH A REAL ROOT EXISTS'
      READ*,A,B
      COUNT=COUNT+1
      IF(F(A)*F(B).GT.0) THEN
      PRINT*, 'NOT POSSIBLE TO FIND THE ROOT IN THE GIVEN INTERVAL'
      STOP
      ENDIF
      PRINT*, 'NOT LIES IN (',A,B,')'
      GOTO 10
      ENDIF
      X=(A+B)/2.0
20    IF (ABS(DF(X)).LT.0.0001)THEN
      PRINT*, 'METHOD FAILS'
      STOP
      ENDIF
      XNEW=X-(F(X)/DF(X))
      IF (ABS(XNEW-X).LT.0.00001) GOTO 50
      X=XNEW
      GOTO 20
50    PRINT*, 'THE REQUIRED ROOT IS:', XNEW
      STOP
      END
```

4.1.1.4 Write a program to find a real root of a polynomial equation by Quotient-difference method. Test the program taking the example $x^2 - 7x^2 + 10x - 2 = 0$.

```
      DIMENSION A(20), D(20, Q(20)
      PRINT*, 'ENTER THE DEGREE OF THE EQUATION'
      READ*,N
      PRINT*,'ENTER ALL COEFFICIENTS FROM THE HIGHEST DEGREE TERM'
```

```
      READ*,A0,(A(I),I=1,N)
      PRINT*,'ENTER MAXIMUM NO. OF ITERATION'
      READ*,MAXIT
      K=0
C     INITIALIZATION
      Q(1)=-A(1)/A0
      DO 10 I=2,N
      Q(I)=0.0
10    CONTINUE
      D0=0
      D(N)=0
      DO 20 I=1, N-1
      D(I)=A(I+1)/AI)
20    CONTINUE
60    ID=0
      DO 30 I=1, N-1
      IF(ABS(D(I)).LT.1E-2) THEN
      ID=1
      ELSE
      ID=0
      GOTO 32
      ENDIF
30    CONTINUE
32    IF(ID.EQ.1) GOTO 70
C     K-TH IERATION
      DO 40 I=1,N
      IF(I.EQ.1)THEN
      Q(I)=Q(I)+D(I)-D0
      ELSE
      Q(I)=Q(I)+D(I)-D(I-1)
      ENDIF
```

```
40    CONTINUE
      D0=0
      D(N)=0
      DO 50 I=1,N-1
      D(I)=Q(I+1) *D(I)/Q(I)
50    CONTINUE
      K=K+1
      IF (K.LT.MAXIT) GOTO 60
70    PRINT*,'.....ROOTS ARE .....'
      PRINT*,(Q(I),I=1,N)
      PRING*,'AT THE ITERATION: ',K
      STOP
      END
```

4.1.2 SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

4.1.2.1 Write a program to find all roots of a system of linear equations by Gauss Elimination method. Test using the following:

$2x + y + z = 10, 3x + 2y + 3z = 18, x + 4y + 9z = 16$

```
C     TO FIND THE VALUE OF UNKNOWN USING GAUSS ELIMINATION
      DIMENSION A(10,10),B(10),X(10)
      PRINT*,'ENTER THE ORDER OF THE COEFFI MATRIX'
      READ*,N
      PRINT*,'ENTER THE MATRIX IN ROWISE SUCH THAT'
      PRINT*,'MAIN DIAGONAL ELEMENTS ARE NON-ZERO'
      READ*,(A(I,J),J=1,N),I=1,N)
      PRINT*,'ENTER THE REQUIREMENT VECTOR'
      READ*, (B(I),I=1,N)
      DO 20 K=1, N-1
      DO 20 I=K+1,N
      IF (A(K,K).EQ.0) THEN
      PRINT*,'METHOD FAILS'
      STOP
```

```
      ENDIF
      RATIO=A(I,K)/A(K,K)
      DO 20 J=1,N
      A(I,J)=A(I,J)-RATIO*A(K,J)
      B(I)=B(I)-RATIO*B(K)
20    CONTINUE
      X(N) = B(N)/A(N,N)
      DO 30 K=N-1,1,-1
      X(K)=B(K)
      DO 40 J=K+1,M
      X(K)=X(K)-A(K,J) *X(J)
40    CONTINUE
      X(K)=X(K)/A(K,K)
30    CONTINUE
      DO 35 I=1,N
      PRINT*, 'X(',I,')='X(I)
35    CONTINUE
      STOP
      END
```

4.1.1.2 Write a program to find all roots of a system of linear equations by Gauss Seidal method. Test using the following:

$$10x-2y-z-w=3, -2x+10y-z-w=15,$$

$$-x-y+10z-2w=27, -x-y-2z+10w=-9$$

```
C     GAUSS SEIDAL METHOD
      DIMENSION X(10),Y(10),A(10,11)
      PRINT*, 'ENTER THE NUMBER OF VARIABLE'
      READ*,N
      PRINT*, 'ENTER THE AUGMENTED MATRIX'
      DO 10 I=1,N
10    READ*, (A(I,J),J=1,N+1)
      PRINT&, 'ENTER THE ACCURACY'
      READ*, EPS
```

```
      DO 6 I=1, N
      X(I)=0.0
      Y(I)=0.0
6     CONTINUE
      ITER=0
15    ITER=ITER+1
      DO 8 I=1,N
      X(I)=A(I,N+1)
      DO 7 J=1,N
      IF(I.EQ.J) GOTO 7
      X(I)=X(I)-A(I,J) *X(J)
7     CONTINUE
      IF(A(I,I).EQ.0) THEN
      PRINT*, 'METHOD FAILS'
      STOP
      ENDIF
      X(I)=X(I)/A(I,I)
8     CONTINUE
      DO 12 K=1,N
      IF (ABS(X(K)-Y(K)), GT.EPS) THEN
      PRINT*, ITER
      DO 13 I=1,N
      Y(I)=X(I)
      PRINT*,X(I)
13    CONTINUE
      GOTO 15
      ENDIF
12    CONTINUE
      PRINT*, 'THE FINAL RESULT IS'
      DO 20 I=1,N
20    PRINT*, 'X('I,')= ',X(I)
      STOP
      END
```

4.1.2.3 Write a program to find the solution of tri-diagonal equations. Text using the following:

$2x+y=2,\ x+2y+z=2,\ y+2z+w=2,\ z+2w=1$

```
      DIMENTION A(5,5)B(5),X(5)
      PRINT*, 'ENTER THE ORDER OF THE COEFFICIENT MATRIX:'
      READ*,N
      PRINT*, 'ENTER THE COEFFICIENT MATRIC ROW-WISE'
      DO 10 I=1,N
      READ*, (A(I,J), J=1,N)
10    CONTINUE
      PRINT*, 'ENTER THE REQUIREMENT VECTOR'
      READ*, (B(I), I=1,N)
      DO 20 I=2,N
      IF (A(I-1,I-1).EQ.0) THEN
      PRINT*, 'TRIDIAGONAL SYSTEM FAILED'
      GOTO 40
      ENDIF
      A(I,I)=(I,I)-A(I,I-1) *A(I-1,I)/A(I-1,I-1)
      B(I)=B(I)-B(I-1) *A(I,I-1)/A(I-1,I-1)
20    CONTINUE
      X(N)=B(N)/A(N,N)
      DO 30 I=N-1,1,-1
      X(I)=(B(I)-A(I,I+1) *X(I+1))/A(I,I)
30    CONTINUE
      PRINT*, (X(I),I=1,N)
40    STOP
      END
```

4.1.3 INTERPOLATION

4.1.3.1 Write a program to find a value of a function by Language Interpolation Technique. Using this to find the following: evaluate $\sqrt{155}$ with the following table

*	150	152	154	156

$y=\sqrt{x}$	12.247	12.329	12.410	12.490

```
C     DETERMINATION OF A FUNCTIONAL VALUE BY LAGRANGE FORMULA
      DIMENSION X(10),P(10),F(10)
      PRINT*, 'ENTER THE NO. OF POINTS (X,Y)'
      READ*, N
      PRINT*, 'ENTER X-VALUES AND Y-VALUES'
      READ*, (X(I), F(I), I=1,N)
      PRINT*, 'ENTER THE VALUE OF X TO CALCULATE OF F(X)'
      READ*, XG
      DO 20 K=1,N
      P(K)=1
      DO 25 I=1,N
      IF (I.EQ.K) GOTO 25
      P(K)=P(K)*(XG-X(I))/(X(K)-X(I))
25    CONTINUE
20    CONTINUE
      FX=0.0
      DO 30 I=1,N
30    FX=FX+P(I)*F(I)
      PRINT*, 'AT X=', XG, 'F(X)=', FX
      STOP
      END
```

4.1.3.2 Write a program to find a value of a function by Newton Forward Interpolation Technique. Using this to find the following: evaluate sin(0.175) with the following table

x	0.15	0.17	0.19	0.21	0.23
sin x	0.14944	0.16918	0.18886	0.20846	0.22798

```
C     DETERMINATION OF A FUNCTIONAL VALUE BY NEWTONS FORWARD
      DIMENSION X(20), F(20), DF(20, 20), DF1(20)
      PRINT*, 'ENTER NUMBER OF POINTS'
      READ*, N
      PRINT*, 'ENTER THE VALUES OF X(I) AND Y(I)'
```

```
      DO 10 I=1,N
10    READ*, X(I), F(I)
      PRINT*, 'ENTER THE VALUE OF X:'
      READ*, XG
      U=(XG-X(1)(1))/(X(2)-X(1))
      SUM=F(1)
      DO 20 I=1,N-1
      DO 20 J=1,N-1
      DF(I,J)=F(J+1)-F(J)
      F(J)=DF(I,J)
      DF1(I)=DF(I,1)
20    CONTINUE
      DO 30 I=1,N-1
      PROD=1
      DO 40 J=1, I
40    PROD=PROD*(U-J+1)/J
      TERM=PROD*DF1(I)
30    SUM=SUM+TERM
      PRINT*, 'CORRESPONDING VALUE OF Y:'
      PRINT*, SUM
      STOP
      END
```

4.1.4 INTEGRATION

4.1.4.1 Write a program to find the value of $\int_a^b f(x)\,dx$ by Trapezoidal Rule. Using the following example test the program: $\int_0^1 \frac{1.0}{1.0+x}\,dx$

```
C     INTEGRATION BY TRAPEZOIDAL RULE
C     F(X)=SQRT(1-0.162*SIN(X)*SIN(X))
      F(X)=X
      PRINT*, 'ENTER THE LIMITS OF INTEGRATION AND NUMBER OF GAP'
```

```
      READ*, A,B,N
      H=(B-A)/N
      SUM=0.0
      DO 20 I=1, N-1
      X=A+I*H
20    SUM=SUM+2*F(X)
      SUM=SUM+F(A)+F(B)
      SUM=H*SUM/2
      PRINT*, 'VALUE OF INTEGRATION=',SUM
      STOP
      END
```

4.1.4.2 Write a programme to find the value of $\int_a^b f(x)dx$ Simpson's 1/3 - Rule. Using the following example test the program: $\int_0^1 \frac{1.0}{1.0+x}dx$

```
C     INTEGRATION BY SYMPSONS 1/3 RULE
C     F(X) = SQRT(1-0.162*SIN(X)*SIN(X))
      F(X)=X
      PRINT*, 'ENTER THE LIMITS OF INTEGRATION'
      READ*, A,B
10    PRINT*, 'ENTER THE NO. OF SUB-INTERVALS'
      READ*,N
      IF (N.NE.N/2*2)THEN
      PRINT*, 'N MUST BE EVEN'
      GOTO 10
      ENDIF
      H=(B-A)/N
      SUN=0.0
      DO 20 I=1,N-1,2
      X=A+I*H
20    SUM-SUM+4*F(X)
      DO 30 I=2, N-2,2
```

```
      X=A+I+H
   30 SUM=SUM+2*F(X)
      SUM=SUM+F(A)+F(B)
      SUM=H*SUM/3.
      PRINT*, 'VALUE OF INTEGRATION =', SUM
      STOP
      END
```

4.1.5 SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

4.1.5.1 Write a program to solve a differential equation by Euler's method and take the following example.

Given $\frac{dy}{dx} = y - x$, where $y(0) = 2$, find $y(0,2)$.

```
C     DIFF EQUA SOLVE BY EULER'S METHOD
      F(X,Y)=(Y-X) / (Y+X)
      PRINT*, 'ENTER INITIAL VALUES (X0, Y0)'
      READ*, X0, Y0
      PRINT*, 'ENTER THE SPACING:'
      READ*,H
      PRINT*, 'ENTER X-VALUE:'
      READ*,X
      PRINT*, '......X AND Y VALUES ARE ......'
   10 YN=Y-+H*F(X0,Y0)
      Y0=YN
      X0=X0+H
      PRINT*, 'X=', X0, 'Y=', YN
      IF (X0.LE.X) GOTO 10
      STOP
      END
```

4.1.5.2 Write a program to solve a differential equation by Milne's Prediction Corrector method and take the following example. Given

$\frac{dy}{dx} = \frac{1}{10}\left(x^2 + y^2\right)$, where $y(0) = 2$, find $y(0,6)$, taking $h = 0.1$.

```
C     MILNE'S PREDICTOR CORRECTOR METHOD
      DIMENSION X(20), Y(20)
      F(A,B)=(1+A*A)*B
      WRITE(*,') 'ENTER THE VALUES FOR X0,Y0,H,N'
      READ*, X(1),Y(1),H,N
      DO 5 I=1,3
      X(I+1)=X(I)+H
      Y(I+1)=Y(I)+H*F(X(I),Y(I))
5     CONTINUE
      WRITE (*,*) 'X              Y'
      DO 7 I=4,N
      X(I+1)=X(I)+H
      YOLD=Y(I-3)+4.0*H/3.0*(2.0*F(X(I-2),Y(I-2)
     * -F(X(I-1),Y(I-1))+2.0*F(X(I),Y(I)))
10    YNEW=Y(I-1)+H/3.0*(F(X(I-1),Y(I-1))+4.0*F(X(I),Y(I))
     * +F(X(I+1),YOLD))
      IF(ABS(YOLD-YNEW).LT.0.00005) GOTO 14
      YOLD=YNEW
      GOTO 10
14    Y(I+1)=YNEW
      PRINT*,X(I+1),Y(I+1)
7     CONTINUE
      STOP
      END
```

4.2 PROBLEMS ON SIMPLE FREQUENCY DISTRIBUTION

4.2.1 Write a program to find mean, median and standard deviation for discrete distribution. Test using following **data**: 34, 12, 56, 18, 22, 45, 35

```
C     MEAN, SD, MEDIAN
      REAL MEAN, MEDIAN, X(10)
      PRINT*, 'ENTER HOW MANY NO.:'
      READ*,N
      PRINT*, 'ENTER THE DATA OF X:'
```

```
      READ*, (X(I),I=1,N)
      SUM=0.0
      DO 30 I=1,N
      SUM=SUM+X(I)
   30 CONTINUE
      MEAN=SUM/N
      SUM1=0.0
      DO 40 I=1,N
      SUM1=SUM1+(X(I)-MEAN)**2
   40 CONTINUE
      VAR=SUM1/N
      SD=sqrt (VAR)
      DO 20 I=1,N-1
      DO 22 J=1,N-1
      IF (X(J).LT.X(J+1) GOTO 22
      T=X(J)
      X(J)=X(J+1)
      X(J+1)=T
   22 CONTINUE
   20 CONTINUE
   50 PRINT*, 'THE SORTED NUMBERS ARE:'
      PRINT*, (X(I),I=1,N)
      IF (N/2*2.EQ.N) THEN
      median=(x(n/2+1)+x(n/2))/2
      else
      median=x(n/2+1)
      endif
      print*,'------THE RESULTS ARE------'
      print*, 'mean=', mean
      print*, 'sd=',sd
      print*,'median=',median
      stop
      end
```

4.2.2 Write a program to find out a correlation coefficient for a set of points $(x_i, y_i), i = 1, 2, ..., n$ and find it using following data:

X	65	66	67	67	68	69	70	72
Y	67	68	65	68	72	72	69	71

```
C     PROGRAM FOR CORRELATION COEFFICIENT
      DIMENSION X(50),Y(50)
      PRINT*, 'ENTER THE SAMPLE SIZE'
      READ*,N
      PRINT*, 'ENTER SAMPLE (X,Y)'
      READ*, (X(I),Y(I),I=1,N)
      SX1=0
      SX2=0
      SY1=0
      SY2=0
      SXY=0
      DO 30 I=1,N
      SX1=SX1+X(I)
      SX2=SX2+X(I) *X(I)
      SY1=SY1+Y(I)
      SY2=SY2+Y(I) *Y(I)
30    SXY=SXY+X(I) *Y(I)
      COV=SXY/N-SX1*SY1/(N**2)
      SDX=SQRT(SX2/N-(SX1/N)**2)
      SDY=SQRT(SY2/N-(SY1/N)**2)
      R=COV/(SDX*SDY)
      PRINT*, 'CORELATION CO EFFICIENT IS:', R
      STOP
      END
```

4.2.3 Write a program for finding a regression line of Y on X for a set of points $(x_1, y_1), i = 1, 2, ..., n$. Using following set of points, test it.

X	4	6	8	10	12
Y	13.72	12.90	12.01	11.14	10.31

```
C     REGRESSION LINE
      DIMENSION X(50),Y(50)
      CHARACTER SIGN
      PRINT*, 'ENTER SAMPLE SIZE'
      READ*,N
      PRINT*, 'ENTER X-VALUES'
      READ*, (X(I),I=1,N)
      PRINT*, 'ENTER Y-VALUES'
      READ*, (Y(I),I=1,N)
      SX1=0
      SX2=0
      SY1=0
      SY2=0
      SXY=0
      DO 10 I=1,N
      SX1=SX1+X(I)
      SX2=SX2+X(I) *X(I)
      SY1=SY1+Y(I)
      SY2=SY2+Y(I) *Y(I)
      SXY=SXY+X(I) *Y(I)
10    CONTINUE
      R=(N*SXY-SX1*SY1)/SQRT(N*SX2-SX1)*(N*SY2-SY1*SY1))
      BYX=(N*SXY-SX1*SY1) / (N*SX2-SX1*SX1)
      SXY=(N*SXY-SX1*SY1) / (N*SY2-SY1*SY1)
      PRINT*, 'CORRELATION COEFFICIENT=',R
      C1=SY1/N-BYX*SX1/N
      SIGN='+'
      IF (C1,LT.0) SIGN='-'
      PRINT (C1.LT.0) SIGN='-'
      PRINT 60, BYX,SIGN,ABS(C1)
60    FORMAT ('REGRESSION LINE OF Y ON X IS:',
      1X, 'Y=',F8.3, 'X',A1,F8.3)
      STOP
      END
```

4.3 Searching and Sorting

4.3.1 Write a program to search a number from a list of numbers by Linear Search Technique.

```
C     SEARCHING OF A NUMBER BYLINEAR SEARCH TECHNIQUE
      DIMENSION A(10)
      PRINT*, 'ENTER NO. OF ELEMENTS'
      READ*, N
      PRINT*, 'ENTER ALL ELEMENTS'
      READ*, (A(I), I=1,N)
      PRINT*, 'ENTER SEARCHING ELEMENT'
      READ*, X
      ID=0
      DO 10 I=1,N
      IF(A(I).EQ.X)THEN
      PRINT*,X, 'IS IN THE LIST IN THE POSITION',I
      ID=1
      ENDIF
10    CONTINUE
      IF (ID.EQ.0) PRINT*,X, 'IS NOT IN THE LIST'
      STOP
      END
```

4.3.2 Write a program to arrange a list of numbers in ascending orders by Bubble sort technique.

```
C     PROGRAM TO ARRANGE THE NUMBERS IN ASSENDING ORDER
      DIMENSIN A(100)
      PRINT*, 'HOW MANY NUMBERS TO BE ARRANGE'
      READ*,N
      PRINT*, 'ENTER THE NUMBERS'
      READ*, (A(I), I=1,N)
```

```
      DO 10 I=1,N-1
      DO 10 J=1, N-1
      IF (A(J).GT.A(J+1)THEN
      TEMP=A(J)
      A(J)=A(J+1)
      A(J+1)=TEMP
      ENDIF
10    CONTINUE
      PRINT*, 'THE ARRANGING NUMBERS ARE:'
      PRINT*, (A(I),I=1,N)
      STOP
      END
```

4.3.3 Write a program to arrange a list of numbers in ascending order by insertion sort technique.

```
C     NUMBERS IN DESENDING ORDER BY INTERSECTION SORT ALGORITHM
      DIMENSION A (25)
      PRINT*, 'HOW MANY NO. ARE ARRANGE'
      READ*, N
      PRINT*, 'ENTER ALL ELEMENTS'
      READ*,(A(I),I=1,N)
      DO 20 I=2, N
      J=I-1
      TEMP=A(I)
10    IF (TEMP.LT.A(J))THEN
      A(J+1)=A(J)
      J=J-1
      IF(J.EQ.0)THEN
      GOTO 15
      ELSE
      GOTO 10
      ENDIF
      ENDIF
15    A(J+1)=TEMP
```

```
20    CONTINUE
      PRINT*, 'SORTED ELEMENTS ARE'
      PRINT*, (A(I),I=1,N)
      STOP
      END
```

4.4 String Manipulation

4.4.1 Write a program to find a number of non-blank characters in a given string

```
C     TO DETERMINE THE NON BLANK CHARACTERS IN A STRING
      CHARACTER S*20
      PRINT*, 'ENTER A STRING:'
      READ(*,5) S
5     FORMAT(A)
      KOUNT=0
      DO 10 I=1,20
      IF (S(I:I) .NE. ' ') KOUNT=KOUNT+1
10    CONTINUE
      WRITE (*,15) KOUNT
15    FORMAT ('NO. OF NON BLANK CHARACTERS ARE',I2)
      STOP
      END
```

4.4.2 Write a program to reverse a string

```
C     TO REVERT A STRING
      CHARACTER S*20, REV*20
      WRITE (*,*) 'ENTER A STRING'
      READ (*,1) S
1     FORMAT (A)
      DO 10 I=1,20
10    REV (20-I+1:20-I+1)=S(I:I)
      WRITE (*,2) REV
2     FORMAT (IX, 'REVERSE IS',A20)
      STOP
      END
```

4.4.3 Write a program to print all characters ofr a string as a triangle.

```
      CHARACTER S*20
      WRITE(*,*) 'ENTER THE STRING'
      READ (*,1) S
1     FORMAT(A20)
      KOUNT=0
      DO S I=1,20
5     IF (S(I:I).NE. '  ') KOUNT=KOUNT+1
      DO 10 I=1, KOUNT
10    WRITE (*,2) S(1:I)
2     FORMAT (1X,A)
      STOP
      END
```

4.4.4 Write a program to print all characters of a string vertically.

```
C     TO ARRANGE A STRING AS A VERTICAL
      CHARACTER S*20
      WRITE (*,*) 'ENTER THE STRING)
      READ (*,1) S
1     FORMAT (A20)
      DO 10 I=1,20
10    WRITE (*,2) S(I:1)
2     FORMAT (1X,A)
      STOP
      END
```

4.4.5 Write a program to search a given character in a given strings.

```
C     SEARCHING FOR A CHARACTER
      CHARTER S*10,CH
      WRITE (*,*) 'ENTER A STRING OF MAXIMUM LENGTH 10'
      READ (*,5) S
5     FORMAT (A10)
```

```
      PRINT*, 'ENTER THE SEARCHING CHARACTER'
      READ (*,6) CH
6     FORMAT (A1)
      DO 10 I=1,10
      IF (S(I:I).EQ.CH) THEN
      PRINT*, CH, 'APPEARS IN THE STRING'
      STOP
      ENDIF
10    CONTINUE
      PRINT*, CH, 'DOES NOT APPEAR'
      STOP
      END
```

4.5 Miscellaneous Problems

4.5.1 Write a program to find the value of determinant

```
C     FOR DETERMINANT
      DIMENSIN A(10,10),B(10,10)
      PRINT*, 'ENTER THE ORDER OF THE MATRIX'
      READ*,N
      PRINT*, 'ENTER THE MATRIX IN ROWISE'
      READ*, ((A(I,J),J=1,N),I=1,N)
      SG=1
      DO 20 K=1, N-1
      LARGE=ABS (A(K,K))
      J=K
      DO 10 I=K+1,N
      IF (LARGE.LT.ABS(A(I,K))) THEN
      LARGE=ABS (A(I,K))
      J=I
      ENDIF
10    CONTINUE
```

```
      IF (LARGE.EQ.0) THEN
      PRINT*, 'METHOD FAILS'
      STOP
      ENDIF
      IF (J.NE.K) THEN
      SG=-SG
      DO 15 I=1,N
      TEMP=A(J,I)
      A(J,I)=A(K,I)
      A(K,I)=TEMP
15    CONTINUE
      ENDIF
      DO 30 I=K+1,N
      RATIO=A(I,K)/A(K,K)
      DO 40 J=1,N
      A(I,J)=A(I,J)-RATIO*A(K,J)
40    CONTINUE
30    CONTINUE
20    CONTINUE
      DET=SG
      DO 50 I=1,N
            DET=DET*A(I,I)
50    CONTINUE
      PRINT*, 'DET=', DET
      STOP
      END
```

4.5.2 Write a program to generate all prime numbers between two specified numbers

```
C     PRIME NUMBER GENERATION AND THEN ADDITION
      INTEGER U, SUM
      PRINT*, 'ENTER LOWER AND UPPER LIMITS:'
      READ*, L,U
```

```
      SUM=0
      PRINT*, '................. PRIME NUMBER... ................
      DO 5 N=L,U
      IF (N.EQ.2)THEN
      SUM=SUM+N
      PRINT*,N
      ELSEIF(N.NE.1)THEN
      K=0
      DO 10 I=2,N/2
      IF(N-I*(N/I).EQ.0) THEN
      K=1
      ENDIF
10    CONTINUE
      IF (K.EQ.0) THEN
      SUM=SUM+N
      PRINT*,N
      ENDIF
      ENDIF
5     CONTINUE
      PRINT*, 'SUM OF ALL PRIME NO. IS: ', SUM
      STOP
      END
```

4.5.3 Write a program to generate first some Fibonacci numbers

```
C     GENERATION OF FIBONACCI NUMBERS
      PRINT*, 'HOW MENY FIBONACCI NUMBERS DO YOU WANT? '
      READ*, N
      N0=0
      N1=1
      PRINT*, '.........FIBONACCI NUMBERS........'
      IF(N.EQ.1) THEN
      PRINT*,N0
      STOP
```

```
      ELSEIF (N.EQ.2) THEN
      PRINT 20, N0,N1
20    FORMAT (10X,I2,/,10X,I2)
      STOP
      ENDIF
      PRINT 20, N0, N1
      KOUNT=2
10    N2=N0+N1
      IF(KOUNT.GE.N) GOTO 50
      PRINT*,N2
      N0=N1
      N1=N2
      KOUNT=KOUNT+1
      GOTO 10
50    STOP
      END
```

4.5.4 Write a program to find the real roots of a quadratic equation.

The roots of the quadratic equation $ax^2 + bx + c = 0$, where a,b and c are real and a is not equal to zero, are given by

$$\left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$$

.The roots are equal when the discriminant $b^2 - 4ac \geq 0$. If $b^2 - 4ac \geq 0$, the roots are real and equal.

```
C     ROOT OF QUADRATIC EQUATION
10    PRINT*, 'ENTER THE COEFFICIENT OF THE EQUATION'
      READ*, A,B,C
      IF(A.EQ.0) THEN
      PRINT*, 'NOT QUADRATIC'
      GOTO 10
      ENDIF
      D=B*B-4, *A*C
      P=-B/(2.*A)
      Q=SQRT(ABS(D))/(2.*A)
      IF(D.LT.0.0) THEN
      PRINT*, 'ROOT ARE COMPLEX'
```

```
      PRINT*, 'ROOT1=', CMPLX(P,Q)
      PRINT*, 'ROOT2=', CMPLX(P,-Q)
      ELSEIF(D.EQ.0.0) THEN
      PRINT*, 'ROOT ARE REAL AND EQUAL'
      PRINT*, 'ROOTS=',P,P
      ELSE
      PRINT*, 'ROOTS ARE REAL AND UNEQUAL'
      R1=P+Q
      R2=P-Q
      PRINT*,'ROOT1=',R1
      PRINT*,'ROOT2=',R2
      ENDIF
      STOP
      END
```

4.5.5 Write a program to find the inverse of a matrix.

```
C     FOR MATRIX INVERSION
      DIMENSION A(10,10), B(10,10)
      WRITE (*,*) 'GIVE THE ORDER OF THE SQUARE MATRIX'
      READ (*,10)N
   10 FORMAT (I2)
      WRITE (*,*) 'ENTER THE ELEMENTS OF THE MATRIX ROW-WISE'
      WRITE(*,*) 'SUCH THAT ALL DIAGONAL ELEMENTS ARE NONZERO'
      DO 20 I=1,N
      READ*, (A(I,J), J=1,N)
   20 CONTINUE
      DO 17 J=1,N
      IF (I.NE.J) THEN
      B(I,J)=0.0
      ELSE
      B(I,J)=1.0
      ENDIF
   17 CONTINUE
```

```
16  CONTINUE
    DO 24 K=1,N
    DO 23 I=1,N
    IF(I.EQ.K) GOTO 23
    IF(A(K,K).EQ.0) THEN
    PRINT*, 'METHOD FAILS'
    STOP
    ENDIF
    R=A(I,K)/A(K,K)
    DO 22 J=1,N
    A(I,J)=A(I,J)-R*A(K,J)
    B(I,J)=B(I,J)-R*B(K,J)
22  CONTINUE
23  CONTINUE
24  CONTINUE
    DO 26 I=1,N
    DO 25 J=1,N
    B(I,J)=B(I,J)/A(I,I)
25  CONTINUE
26  CONTINUE
    DO 28 I=1,N
    WRITE(*,*) (B(I,J),J=1,N)
28  CONTINUE
    STOP
    END
```

4.5.6 Write a program to find the value of sin(x)

```
C   DETERMINATION OF VALUE OF SINE SERIES
60  FORMAT (10X,1HX,5X,6HSIN(X))
    DO 10 XD=0, 90, 15
    X=XD*3.14159/180
    N=1
```

```
      TERM=X
      SUM=X
15    TERM=TERM* (-X*X/((2.0*N)*(2.0*N+1)))
      IF (ABS(TERM).LT.1.0E-3) GOTO 40
      SUM=SUM+TERM
      N=N+1
      GOTO 15
40    PRINT 50,XD,SUM
50    FORMAT (5X,F6.2,5X,F6.2)
10    CONTINUE
      STOP
      END
```

4.5.7 Write a program to find the values of cos(x)

```
C     DETERMINATION OF VALUE OF COSINE SERIES
      RINT 60
60    FORMAT (10X,1HX,5X,6HCOS(X))
      DO 10 XD=0, 90, 15
      X=XD*3.14159/180
      N=1
      TERM=1
      SUM=1
15    TERM=TERM* (-X*X/((2.0*N-1)*(2.0*N)))
      IF(ABS(TERM).LT.1.0E-3)GOTO 40
      SUM=SUM+TERM
      N=N+1
      GOTO 15
40    PRINT 50,XD,SUM
50    FORMAT (5X,F6.2,5X,F6.2)
10    CONTINUE
      STOP
      END
```

4.5.8 Write a program to find the value of ^nC_r using function sub-program.

```
C     SUBPROGRAM TO CALCULATE NCR
      FUNCTION FACT (N)
      PROD=1
      IF(N.EQ.0) THEN
      FACT=PROD
      ELSE
      DO 10 I=1,N
      PROD=PROD*I
10    CONTINUE
      FACT=PROD
      ENDIF
      RETURN
      END
C     MAIN PROGRAM
      INTEGER R
      PRINT*, 'ENTER THE VALUE OF N'
      READ*,N
      PRINT*, 'ENTER THE VALUE OF R'
      READ*, R
      NCR=FACT(N)/(FACT(N-R)*FACT(R))
      PRINT*, 'THE NCR IS=',NCR
      STOP
      END
```

4.5.9 Write a program to find the largest and smallest numbers amongst a given list of numbers using subroutine subprogram through COMMON statement

```
C     DETERMINATION
C     MAIN PROGRAM
      COMMON X(10), LARGE, SMALL, N
      PRINT*, 'ENTER NO. OF ELEMENTS'
      READ*, N
      PRINT*, 'ENTER ALL ELEMENTS'
```

```
      READ*, (X(I), I=1,N)
      CALL LSMALL
      PRINT*, 'THE SMALLEST ELEMENT IS:',SMALL
      PRINT*, 'THE LARGEST ELEMENT IS:',LARGE
      STOP
      END
C     SUBROUTINE
      SUBROUTINE LSMALL
      COMMON A(10),L,S,N
      S=A(1)
      L=A(1)
      DO 10 I=2,N
      IF(S.GT.A(I)) S=A(I)
      IF(L.LT.A(I)) L=A(I)
10    CONTINUE
      RETURN
      END
```

4.5.10 Write a program to multiply two matrices using subroutine subprogram through COMMON statement

```
C     MATRICES MULTIPLICATION USING COMMON STATEMENT
C     MAIN PROGRAM
      COMMON A(10,10),B(10,10),C(10,10)
      INTEGER P,Q
      PRINT*, 'ENTER ORDERS OF A-MATRIX'
      READ*,M,N
      PRINT*, 'ENTER ORDERS OF B-MATRIX'
      READ*, ((A(I,J), J=1,N),I=1,M)
      PRINT*,'ENTER ELEMENTS OF B-MATRIX'
      READ*,((B(I,J),J=1,Q),I=1,P)
      IF(N.EQ.P) THEN
      CALL PROD (M,N,P,Q)
```

```
      PRINT*, 'THE PRODUCT OF TWO MATRICES IS:'
      DO 10 I=1,M
      PRINT*, (C(I,J),J=1,Q)
10    CONTINUE
      ELSE
      PRINT*, 'MULTIPLICATION NOT COMPATIBLE'
      ENDIF
      STOP
      END
C     SUBROUTINE
      SUBROUTINE PROD (M,N,P,Q)
      INTEGER P,Q
      COMMON A(10,10),B(10,10),C(10,10)
      DO 20 I=1,M
      DO 30 J=1,Q
      C(I,J)=0.0
      DO 40 K=1,N
      C(I,J)=C(I,J)+A(I,K)*B(K,J)
40    CONTINUE
30    CONTINUE
20    CONTINUE
      RETURN
      END
```

4.5.11 Write a program to find the area and perimeter of a circle and a triangle.

```
C     TO CALCULATE OF AREA AND PERIMETER OF A TRIANGLE AND A CIRCLE
      PRINT*, 'ENTER THE CHOICE OF SHAPE'
      PRINT*, 'FOR TRIANGLE: ENTER 1'
      PRINT*, 'FOR CIRCLE: ENTER 2'
      READ*, CH
      IF(CH.EQ.1) THEN
      PRINT*, 'ENTER THE SIDES OF TRIANGLE'
      READ*, A,B,C
      IF (A+B.GT.C).AND.(B+C.GT.A).AND.(C+A.GT.B)) THEN
      S=(A+B+C)/2.0
      AREA=SQRT(S*(S-A)*(S-B)*(S-C))
      PERI=A+B+C
      PRINT*,'THE AREA OF THE TRIANGLE= ',AREA
```

```
PRINT*, 'THEPERIMETER OF THE TRIANGLE =',PERI
ELSE
PRINT*, 'TRIANGLE IS NOT POSSIBLE'
ENDIF
ELSELF (CH.EQ.2) THEN
PRINT*, 'ENTER RADIUS OF A CIRCLE'
READ*,R
PI=22.0/7.0
AREA=PI*R*R
PERI=2*PI*R
PRINT*,'THE AREA OF THE CIRCLE ARE:',AREA
PRINT*,'THE PERIMETER OF THE CIRCLE:',PERI
ELSE
PRINT*,'CHOICE IS NOT CORRECT'
ENDIF
STOP
END
```

4.5.12 Write a program to split of a positive integer number into its digits and then to find the sum and product of all digits

```
C    SPLITTING OF A NUMBER INTO ITS DIGIT AND
C    THEN TO FIND SUM AND PRODUCT OF DIGITS
     INTEGER SUM, PROD, DIGIT
     PRINT&,'ENTER THE POSITIVE INTEGER NUMBER'
     READ*,NUM
     SUM=0
     PROD=1
     PRINT*,'....DIGITS.....'
10   IF (NUM.GT.0) THEN
     DIGIT=NUM-(NUM/10)*10
     PRINT*,DIGIT
     SUM=SUM+DIGIT
     PROD=PROD*DIGIT
```

```
      NUM=NUM/10
      GOTO 10
      ENDIF
      PRINT*,'SUM=',SUM
      PRINT*,'PRODUCT=',PROD
      STOP
      END
```

4.5.13 Write a program to find the value of greatest common divisor of two integers

```
C     TO FIND GREATEST COMMON DIVISION OF TWO NUMBERS
      PRINT*,'ENTER TWO NUMBERS'
      READ*,NUM1,NUM2
      IF (NUM2.GT.NUM1)THEN
      K=NUM1
      NUM1=NUM2
      NUM2=K
      ENDIF
5     IF(NUM2.EQ.0)GOTO 10
      K=NUM1-NUM1/NUM2*NUM2
      NUM1=NUM2
      NUM2=K
      GOTO T
10    PRINT*,'THE GCD IS:', NUM1
      STOP
      END
```

4.5.14 Write a program to determine whether a given matrix is symmetric or orthogonal or not.

```
C     TO CHECK WHETHER A MATRIX IS SYMMETRIC AND ORTHOGONAL
      DIMENSION A(10,10),T(10,10),P(10,10)
      PRINT*,'ENTER THE ORDER OF A SQUARE MATRIX'
      READ*,N
      PRINT*,'ENTER THE MATRIX'
      READ*,(A(I,J),J=1,N),I=1,N)
      PRINT*,'ENTER THE CHOICE OF CHECKING'
```

```
      PRINT*,'FOR SYMMETRIC: ENTER 1'
      PRINT*,'FOR ORTHOGONAL: ENTER 2'
      READ*,CH
      IF(CH.EQ.1)THEN
      DO 10 I=1,N
      DO 15 J=1,N
      IF(I.NE.J) THEN
      IF(A(I,J).NE.A(J,I))THEN
      PRINT*,'MATRIX IS NOT SYYMETRIC'
      STOP
      ENDIF
      PRINT*,'MATRIX IS SYMMETRIC'
      ENDIF
15    CONTINUE
10    CONTINUE
      ELSEIF(CH.EQ.2)THEN
      DO 20 I=1,N
      DO 25 J=1,N
      T(I,J)=A(J,I)
25    CONTINUE
20    CONTINUE
      DO 30 I=1,N
      DO 35 J=1,N
      P(I,J)=0
      DO 40 K=1,N
      P(I,J)=P(I,J)+A(I,K)*T(K,J)
40    CONTINUE
35    CONTINUE
30    CONTINUE
      ID=0
      ID1=0
      DO 50 I=1,N
      DO 55 J=1,N
      IF (I.NE.J).AND.(P(I,J).EQ.0.0)) ID=1
```

```
      IF ((I.EQ.J).AND.(P(I,J).EQ.1.0)) ID1=1
55    CONTINUE
50    CONTINUE
      IF((ID.EQ).AND.(ID1.EQ.1))THEN
      PRINT*,'THE MATRIX IS ORTHOGONAL'
      ELSE
      PRINT*,'THEMATRIX IS NOT ORTHOGONAL'
      ENDIF
      ENDIF
      STOP
      END
```

5.0 Unit Summary

In this unit, Solution of algebraic and transcendental equations, Solution of a system of linear equations, Numerical, Integration, Interpolation, Numerical solution of ordinary differential equation, Eigen value problem, Some Statistical methods, some searching and sorting techniques and Some Miscellaneous problems are implemented using FORTRAN-77 programming language.

6.0 Self Assessment Questions

1. Write a program to find the root of a polynomial equation.
2. Write a program to find the root of a partial differential equation by finite difference method.
3. Write a program to find the solution of tri-diagonal equations.
4. Write a program to find the functional value by Splin interpolation.
5. Write a program to find and replace a given letter in a given string.
6. Write a program to convert a name into abbreviated form.
7. Write a program to find all combinations of letters of a word.

7.0 Suggested further readings

1. Balgurusamy E., Programming in ANSI C, Tata McGraw-Hill, 1992.
2. Venugopal K.R., Programming with C, Tata McGraw-Hill, 2001.
3. Kumar R. and Agarwal R., Programming in ANSI C, Tata McGraw-Hill, 1993.
4. Xavier C, Fortran 77 and Numerical Methods, New Age International, 2000.

"Learner's Feed-back"

After going through the Modules / Units please answer the following questionnaire.
Cut the portion and send the same to the Directorate.

To
The Director
Directorate of Distance Education,
Vidyasagar University
Midnapore - 721 102

1. The modules are : (give ✓ in appropriate box)
☐ easily understandable; ☐ very hard; ☐ partially understandable.

2. Write the number of the Modules/Units which are very difficult to understand :
....
....
....

3. Write the number of Modules / Units which according to you should be re-written :
....
....
....

4. Which portion / page is not understandable to you? (mention the page no. and portion)
....
....
....
....

5. Write a short comment about the study material as a learner.
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....
....

....
(Full Signature of the Learner)

Date : ....

Enrolment No. ....

Phone / Mobile No. ....


BADAL DINESH BHAVAN