Chapter 7

Dombi m-polar fuzzy graphs

7.1 Introduction

The t-operators and max & min operators uplifted the fuzzy graph [17, 86, 134] and little effort has been made to use new operators. Triangular co-norms (t-co-norms) and triangular norms (t-norms) were presented by Menger [35]. Alsine et al. [36] have demonstrated that t-conorms and t-norms are models for unification and intersection of fuzzy sets(FSs). Since that time, for the same effect, lots of other researchers have introduced different types of t-operators [37]. Zadeh's conventional T-operators were especially used in FG theory and decision making processes. This chapter defines join and union, composition, cartesian product of two Dombi mPFGs. Some characteristics of isomorphism are discussed as well as self complementary Dombi mPFG.

7.2 Some preliminaries

Definition 7.2.1. *[145] A triangular norm (t-norm) is a binary operation* $T : [0,1]^2 \to [0,1]$ *if it fulfills the following* $\forall\ q, r$ *and* $u \in [0,1]$*:*

1) $T(1, q) = q.\quad$ *(boundary condition)*

2) $T(q, r) = T(r, q).\quad$ *(commutativity)*

3) $T(q, T(r, s)) = T(T(q, r), s).\quad$ *(associativity)*

4) $T(q, r) \le T(q, s)\ \ if\ \ r \le s.\quad$ *(monotonicity)*

Definition 7.2.2. *[145] A triangular conorm (t-conorm) is a binary operation S : $[0,1]^2 \to [0,1]$ if there exists a t-norm T s.t. $\forall\ (s,t) \in [0,1]^2$*
$S(s,t) = 1 - T(1-s, 1-t)$.
Popular choices for t-norms are:

- *The minimum operator M : $M(s,t) = min(s,t)$.*
- *The product operator P : $P(s,t) = st$.*
- *The Lukasiewiczs t-norm W : $W(s,t) = max(s+t-1, 0)$.*

Popular choices for corresponding dual t-conorms are:

- *The maximum operator M^* : $M^*(s,t) = max(s,t)$.*
- *The probabilistic sum P^* : $P^*(s,t) = s+t-st$.*
- *The bounded sum W^* : $W^*(s,t) = min(s+t, 1)$.*

The Dombi family

$$t-norm \quad \frac{1}{1+[(\frac{1-s}{s})^{\mu}+(\frac{1-t}{t})^{\mu}]^{\frac{1}{\mu}}} : \mu > 0$$
$$t-conorm \quad \frac{1}{1+[(\frac{1-s}{s})^{-\mu}+(\frac{1-t}{t})^{-\mu}]^{\frac{1}{-\mu}}} : \mu > 0$$
$$negation \quad 1-s.$$

The Hamacher family

$$t-norm \quad \frac{st}{(1-\mu)(s+t-st)} : \mu > 0$$
$$t-conorm \quad \frac{s+t+(\mu-2)st}{1+(\mu-1)st} : \mu > 0$$
$$negation \quad 1-s.$$

Another set of T-operators is

$$T(s,t) = \frac{st}{s+t-st}$$

,

$$S(s,t) = \frac{s+t-2st}{1-st}$$

which is obtained by taking $\mu = 0$, in the Hamacher family and $\mu = 1$ in the Dombi family of t-norms and t-conorms. Also $P(s,t) \leq \frac{st}{s+t-st} \leq M(s,t)$ and $M^*(s,t) \leq \frac{(s+t-2st)}{(1-st)} \leq P^*(s,t)$.

7.3 Dombi m-polar fuzzy graph

In this section, we defined Dombi m-polar fuzzy graph (Dombi mPFG) and different types of product on Dombi mPFG.

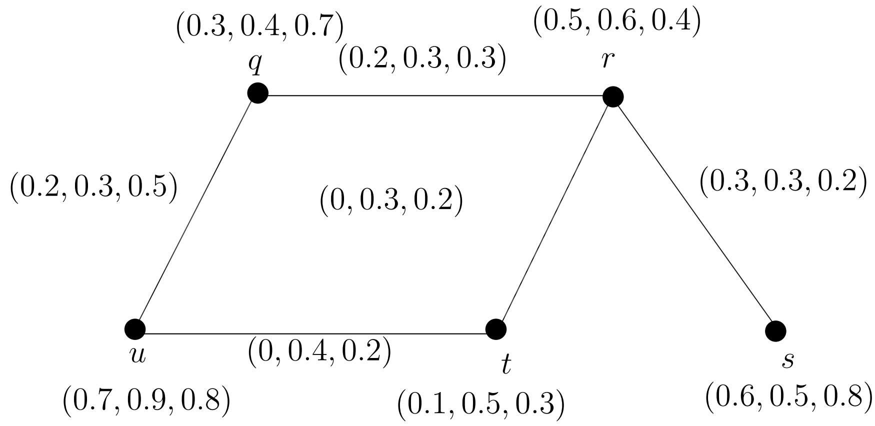

Figure 7.1: Dombi 3PFG G.

Definition 7.3.1. *An ordered pair $G = (C, D)$ is a Dombi mPFG on underlying set V where $C : V \to [0, 1]$ ia a mPFSS in V and $D : V \times V :\to [0, 1]$ is a symmetric mPF relation on A s.t., $p_i \circ D(g, h) \leq \frac{(p_i \circ C(g))(p_i \circ C(h))}{(p_i \circ C(g)) + (p_i \circ C(h)) - (p_i \circ C(g))(p_i \circ C(h))}$, $\forall\ i = 1, 2, \ldots, m$.*

We call D the Dombi mPFES and C the Dombi mPFVS of G.

Example 7.3.1. *In figure 7.1, we consider Dombi mPFG over $V = \{q, r, s, t, u\}$ where $A = \{\frac{q}{(0.3,0.4,0.7)}, \frac{r}{(0.5,0.6,0.4)}, \frac{s}{(0.6,0.5,0.8)}, \frac{t}{(0.1,0.5,0.3)}, \frac{u}{(0.7,0.9,0.8)}\}$ and $B = \{\frac{qr}{(0.2,0.3,0.3)}, \frac{rs}{(0.3,0.3,0.2)}, \frac{rt}{(0,0.3,0.2)}, \frac{qu}{(0.2,0.3,0.5)}, \frac{ut}{(0,0.4,0.2)}\}$.*

7.4 Products on Dombi m-polar fuzzy graphs

In this section, we defined different types of products on Dombi mPFGs G_1 and G_2. These operations are Cartesian product, composition, direct product, semi-strong product and strong product.

7.4.1 Direct product on Dombi m-polar fuzzy graphs

Definition 7.4.1. *Let $G_1 = (V_1, C_1, D_1)$ and $G_2 = (V_2, C_2, D_2)$ be two Dombi mPFGs. The direct product $G_1 \times G_2 = (C_1 \times C_2, D_1 \times D_2)$ of two Dombi mPFGs G_2 and G_1, as it follows $\forall\ i = 1, 2, \ldots, m$,*

i) $p_i \circ (C_1 \times C_2)(s_1, s_2) = \frac{(p_i \circ C_1(s_1))(p_i \circ C_2(s_2))}{(p_i \circ C_1(s_1)) + (p_i \circ C_2(s_2)) - (p_i \circ C_1(s_1))(p_i \circ C_2(s_2))}$
$\forall\ (s_1, s_2) \in V_1 \times V_2$

ii) $p_i \circ (D_1 \times D_2)((s_1, s_2)(t_1, t_2)) = \frac{(p_i \circ D_1(s_1,t_1))(p_i \circ D_2(s_2,t_2))}{(p_i \circ D_1(s_1,t_1)) + (p_i \circ D_2(s_2,t_2)) - (p_i \circ D_1(s_1,t_1))(p_i \circ D_2(s_2,t_2))}$
$\forall\ s_1t_1 \in E_1$ *and* $s_2t_2 \in E_2$

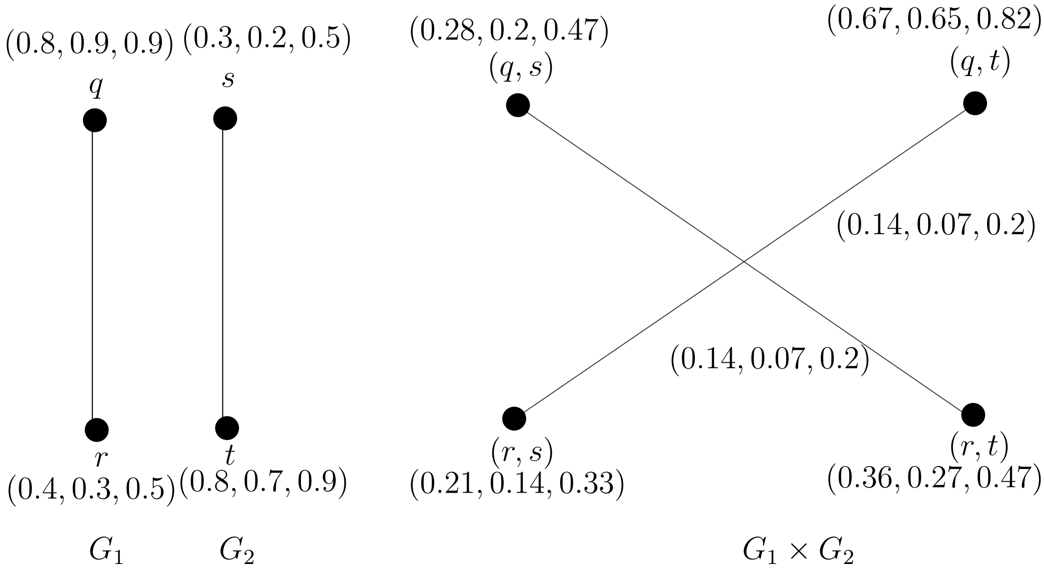

Figure 7.2: Direct product of two Dombi 3PFG G_1 and G_2.

Example 7.4.1. *Consider two Dombi mPFGs* $G_1 = (V_1, C_1, D_1)$ *and* $G_2 = (V_2, C_2, D_2)$ *where,* $C_1 = \{\frac{q}{(0.8,0.9,0.9)}, \frac{r}{(0.4,0.3,0.5)}\}$, $D_1 = \{\frac{qr}{(0.3,0.2,0.4)}\}$ *and* $C_2 = \{\frac{s}{(0.3,0.2,0.5)}, \frac{t}{(0.8,0.7,0.9)}\}$, $D_2 = \{\frac{st}{(0.2,0.1,0.3)}\}$. *Then we have* $(C_1 \times C_2)(q, s) = (0.28, 0.20, 0.47)$, $(C_1 \times C_2)(r, t) = (0.36, 0.27, 0.47)$, $(C_1 \times C_2)(q, t) = (0.67, 0.65, 0.82)$, $(C_1 \times C_2)(r, s) = (0.21, 0.14, 0.33)$, $(D_1 \times D_2)((q, s)(r, t)) = (0.14, 0.07, 0.20)$ *and* $(D_1 \times D_2)((q, t)(r, s)) = (0.14, 0.07, 0.20)$.

Proposition 7.4.1. *Let* G_1 *and* G_2 *be the Dombi mPFGs of the graphs* G_1^* *and* G_2^* *respectively. Then* $G_1 \times G_2$ *is the Dombi mPFG of* $G_1^* \times G_2^*$ *where* $G_1 \times G_2$ *is the direct product of* G_1 *and* G_2.

Proof. Consider $s_1t_1 \in E_1$ and $s_2t_2 \in E_2$. Then $\forall\ i$,

$$
\begin{aligned}
& p_i \circ (D_1 \times D_2)((s_1, s_2)(t_1, t_2)) \\
= \; & \frac{(p_i \circ D_1(s_1, t_1))(p_i \circ D_2(s_2, t_2))}{(p_i \circ D_1(s_1, t_1)) + (p_i \circ D_2(s_2, t_2)) - (p_i \circ D_1(s_1, t_1))(p_i \circ D_2(s_2, t_2))} \\
= \; & T(p_i \circ D_1(s_1, t_1), p_i \circ D_2(s_2, t_2)) \\
\leq \; & T(p_i \circ D_1(s_1, t_1), \frac{(p_i \circ C_2(s_2))(p_i \circ C_2(t_2)}{(p_i \circ C_2(s_2)) + (p_i \circ C_2(t_2) - (p_i \circ C_2(s_2))(p_i \circ C_2(t_2)}) \\
\leq \; & T(\frac{(p_i \circ C_1(s_1))(p_i \circ C_1(t_1))}{(p_i \circ C_1(s_1)) + (p_i \circ C_1(t_1) - (p_i \circ C_1(s_1))(p_i \circ C_1(t_1)}, \\
& \qquad \frac{(p_i \circ C_2(s_2))(p_i \circ C_2(t_2)}{(p_i \circ C_2(s_2)) + (p_i \circ C_2(t_2) - (p_i \circ C_2(s_2))(p_i \circ C_2(t_2)})) \\
& (let,\; p_i \circ C_1(s_1) = x_1,\; p_i \circ C_1(t_1) = x_2,\; p_i \circ C_1(s_2) = y_1,\; p_i \circ C_1(t_2) = y_2) \\
= \; & \frac{\frac{(x_1)(x_2)(y_1)(y_2)}{((x_1)+(x_2)-(x_1)(x_2))(y_1)+(y_2)-(y_1)(y_2))}}{\frac{(x_1)(x_2)}{x_1)+(x_2)-(x_1)(x_2)} + \frac{(y_1)(y_2)}{(y_1)+(y_2)-(y_1)(y_2)} - \frac{(x_1)(x_2)(y_1)(y_2)}{(x_1)+(x_2)-(x_1)(x_2))((y_1)+(y_2)-(y_1)(y_2))}} \\
= \; & \frac{\frac{(x_1)(y_1)(x_2)(y_2)}{(x_1)+(y_1)-(x_1)(y_1))((x_2)+(y_2)-(x_2)(y_2))}}{\frac{(x_1)(y_1)}{(x_1)+(y_1)-(x_1)(y_1)} + \frac{(x_2)(y_2)}{(x_2)+(y_2)-(x_2)(y_2)} - \frac{(x_1)(y_1)(x_2)(y_2)}{(x_1)+(y_1)-(x_1)(y_1))((x_2)+(y_2)-(x_2)(y_2))}} \\
= \; & \frac{(p_i \circ (C_1 \times C_2)(s_1, s_2))(p_i \circ (C_1 \times C_2)(t_1, t_2))}{(p_i \circ (C_1 \times C_2)(s_1, s_2)) + (p_i \circ (C_1 \times C_2)(t_1, t_2)) - (p_i \circ (C_1 \times C_2)(s_1, s_2))(p_i \circ (C_1 \times C_2)(t_1, t_2))}.
\end{aligned}
$$

□

7.4.2 Cartesian product of two Dombi m-polar fuzzy graph

Definition 7.4.2. *Let $G_1 = (V_1, C_1, D_1)$ and $G_2 = (V_2, C_2, D_2)$ be two Dombi mPFGs. The cartesian product $G_1 \Box G_2 = (C_1 \Box C_2, D_1 \Box D_2)$ of two Dombi mPFGs G_2 and G_1 of the graphs $G_2^* = (V_2, E_2)$ and $G_1^* = (V_1, E_1)$, as it follows, for all $i = 1, 2, \ldots, m$*

i) $p_i \circ (C_1 \Box C_2)(r_1, r_2) = \frac{(p_i \circ C_1(r_1))(p_i \circ C_2(r_2))}{(p_i \circ C_1(r_1)) + (p_i \circ C_2(r_2)) - (p_i \circ C_1(r_1))(p_i \circ C_2(r_2))}$, $\forall\ (r_1, r_2) \in V_1 \times V_2$.

ii) $p_i \circ (D_1 \Box D_2)((r, r_2)(r, s_2)) = \frac{(p_i \circ C_1(r))(p_i \circ D_2(r_2 s_2))}{(p_i \circ C_1(r)) + (p_i \circ D_2(r_2 s_2)) - (p_i \circ C_1(r_1))(p_i \circ D_2(r_2 s_2))}$ *for all* $r \in V_1$, $\forall\ r_2 s_2 \in E_2$.

iii) $p_i \circ (D_1 \Box D_2)((r_1, t)(s_1, t)) = \frac{(p_i \circ D_1(r_1 s_1))(p_i \circ C_2(t))}{(p_i \circ D_1(r_1 s_1)) + (p_i \circ C_2(t)) - (p_i \circ D_1(r_1 s_1))(p_i \circ C_2(t))}$ $\forall\ r_1 s_1 \in E_1$, *for all* $t \in V_2$.

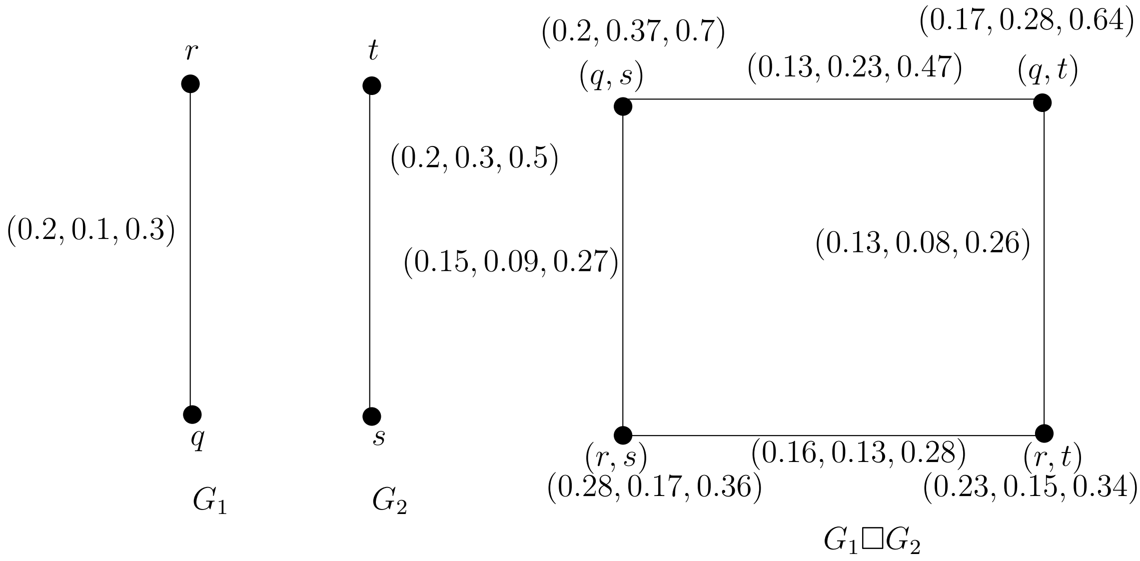

Figure 7.3: Cartesian product of two Dombi 3PFG G_1 and G_2.

Example 7.4.2. *Consider two Dombi mPFGs* $G_1 = (V_1, C_1, D_1)$ *and* $G_2 = (V_2, C_2, D_2)$ *where,* $C_1 = \{\frac{q}{(0.3,0.5,0.9)}, \frac{r}{(0.5,0.2,0.4)}\}$, $D_1 = \{\frac{qr}{(0.2,0.1,0.3)}\}$ *and* $C_2 = \{\frac{s}{(0.4,0.6,0.8)}, \frac{t}{(0.3,0.4,0.7)}\}$, $D_2 = \{\frac{st}{(0.2,0.3,0.5)}\}$. *Then we have*

$D_1 \Box D_2((q, s)(r, s)) = (0.15, 0.09, 0.27)$,

$D_1 \Box D_2((r, s)(r, t)) = (0.23, 0.15, 0.34)$,

$D_1 \Box D_2((r, t)(q, t)) = (0.13, 0.08, 0.26)$,

$D_1 \Box D_2((q, t)(q, s)) = (0.13, 0.23, 0.47)$.

Therefore $G_1 \Box G_2$ *is not a Dombi mPFG.*

Definition 7.4.3. *If mPF membership degree of each of the Dombi mPFG G is come from* $[0, 1]$ *and every vertex in G is crisp, then G is the Dombi mPF edge graph(Dombi mPFEG).*

Proposition 7.4.2. *The cartesian product* $G_1 \Box G_2$ *of* G_1 *and* G_2 *is Dombi mPFG of* $G_1^* \Box G_2^*$, *where* G_2 *and* G_1 *be the Dombi mPFEGs of the graphs* G_2^* *and* G_1^*.

Proof. Consider $r \in V_1$, $r_2t_2 \in E_2$, then

$$\begin{aligned}
& p_i \circ (D_1 \Box D_2)((r,r_2)(r,s_2)) \\
= \; & \frac{(p_i \circ C_1(r))(p_i \circ D_2(r_2,s_2))}{(p_i \circ C_1(r)) + (p_i \circ D_2(r_2,s_2)) - (p_i \circ C_1(r))(p_i \circ D_2(r_2,s_2))} \\
= \; & T(p_i \circ C_1(r), p_i \circ D_2(r_2,s_2)) \\
= \; & T(1, p_i \circ D_2(r_2,s_2)) \\
= \; & p_i \circ D_2(r_2,s_2) \\
\leq \; & \frac{(p_i \circ C_2(r_2))(p_i \circ C_2(s_2))}{(p_i \circ C_2(r_2)) + (p_i \circ C_2(s_2)) - (p_i \circ C_2(r_2))(p_i \circ C_2(s_2))} \\
= \; & \frac{(p_i \circ (C_1 \Box C_2)(R))(p_i \circ (C_1 \Box C_2)(S))}{(p_i \circ (C_1 \Box C_2)(R)) + (p_i \circ (C_1 \Box C_2)(S)) - (p_i \circ (C_1 \Box C_2)(R))(p_i \circ (C_1 \Box C_2)(S))} \\
& \textit{where},\; R = (r,r_2),\; S = (r,s_2).
\end{aligned}$$

Consider $t \in V_2$, $r_1s_1 \in E_1$. Then

$$\begin{aligned}
& p_i \circ (D_1 \Box D_2)((r_1,t)(s_1,t)) \\
= \; & \frac{(p_i \circ D_1(r_1,s_1))(p_i \circ C_2(t))}{(p_i \circ D_1(r_1,s_1)) + (p_i \circ C_2(t)) - (p_i \circ D_1(r_1,s_1))(p_i \circ C_2(t))} \\
= \; & T((p_i \circ D_1(r_1,s_1), p_i \circ C_2(t)) \\
= \; & T(p_i \circ D_1(r_1,s_1), 1) \\
= \; & p_i \circ D_1(r_1,s_1) \\
\leq \; & \frac{(p_i \circ C_1(r_1))(p_i \circ C_1(s_2))}{(p_i \circ C_1(r_1)) + (p_i \circ C_1(s_1)) - (p_i \circ C_1(r_1))(p_i \circ C_1(s_))} \\
= \; & \frac{(p_i \circ (C_1 \Box C_2)(U))(p_i \circ (C_1 \Box C_2)(V))}{(p_i \circ (C_1 \Box C_2)(U)) + (p_i \circ (C_1 \Box C_2)(V)) - (p_i \circ (C_1 \Box C_2)(U))(p_i \circ (C_1 \Box C_2)(V))} \\
& \textit{where},\; U = (r_1,t),\; V = (s_1,t).
\end{aligned}$$

Hence proved. □

Example 7.4.3. *Taking two Dombi mPFG $G_1 = (V_1, C_1, D_1)$ and $G_2 = (V_2, C_2, D_2)$, where $C_1(x) = (1,1,1)\ \forall\ x \in V_1$ and $D_1 = \{\frac{qr}{(0.6,0.7,0.9)}\}$, $C_2(y) = (1,1,1)\ \forall\ y \in V_2$ and $D_2 = \{\frac{st}{(0.5,0.4,0.6)}, \frac{tu}{(0.3,0.5,0.7)}\}$. Then we have,*

$D_1 \Box D_2((q,s)(q,t)) = (0.5, 0.4, 0.6)$,

$D_1 \Box D_2((q,t)(q,u)) = (0.3, 0.5, 0.7)$,

$D_1 \Box D_2((r,s)(r,t)) = (0.5, 0.4, 0.6)$,

$D_1 \Box D_2((r,t)(r,u)) = (0.3, 0.5, 0.7)$,

$D_1 \Box D_2((q,s)(r,s)) = (0.6, 0.7, 0.9)$,

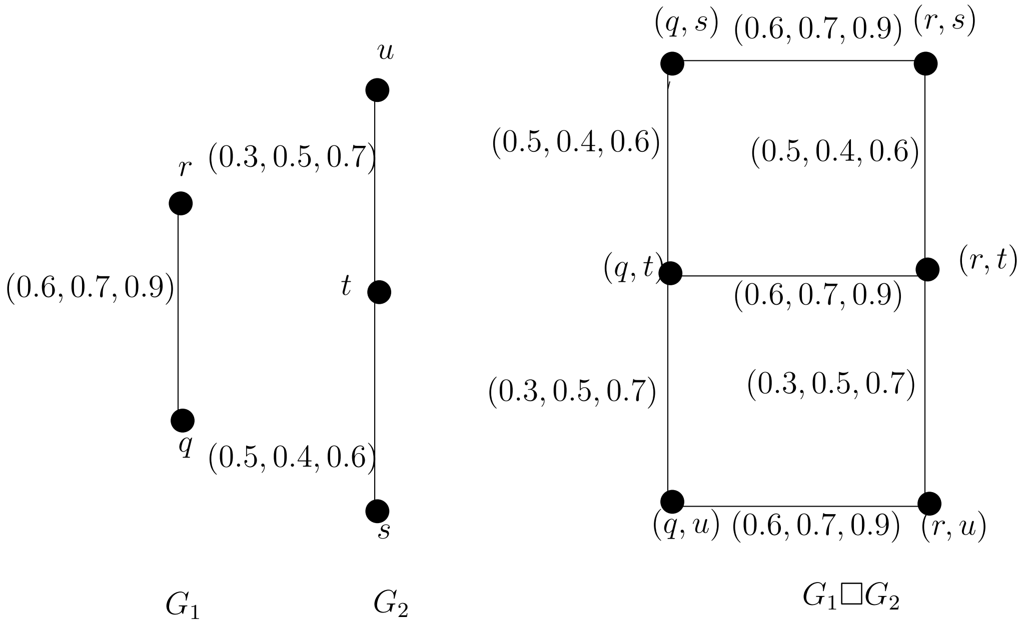

Figure 7.4: cartesian product of two Dombi 3PFEG G_1 and G_2.

$D_1 \Box D_2((q,t)(r,t)) = (0.6, 0.7, 0.9)$,

$D_1 \Box D_2((q,u)(r,u)) = (0.6, 0.7, 0.9)$.

Here we get $G_1 \Box G_2$ is the Dombi mPFEG of $G_1^* \Box G_2^*$.

Definition 7.4.4. *Let $G_1 = (V_1, C_1, D_1)$ and $G_2 = (V_2, C_2, D_2)$ be two Dombi mPFGs. The semi strong product $G_1 \bullet G_2 = (C_1 \bullet C_2, D_1 \bullet D_2)$ of the Dombi mPFGs G_2 and G_1 of $G_2^* = (V_2, E_2)$ and $G_1^* = (V_1, E_1)$ respectively as follows:*

i) $p_i \circ (C_1 \bullet C_2)(s_1, s_2) = \frac{(p_i \circ C_1(s_1))(p_i \circ C_2(s_2))}{(p_i \circ C_1(s_1)) + (p_i \circ C_2(s_2)) - (p_i \circ C_1(s_1))(p_i \circ C_2(s_2))}$, $\forall\ (s_1, s_2) \in V_1 \times V_2$.

ii) $p_i \circ (D_1 \bullet D_2)((s, s_2)(s, t_2)) = \frac{(p_i \circ C_1(s))(p_i \circ D_2(s_2 t_2))}{(p_i \circ C_1(s)) + (p_i \circ D_2(s_2 t_2)) - (p_i \circ C_1(s_1))(p_i \circ D_2(s_2 t_2))}$ $\forall\ s \in V_1$, $\forall\ s_2 t_2 \in E_2$.

iii) $p_i \circ (D_1 \bullet D_2)((s_1, s_2)(t_1, t_2)) = \frac{(p_i \circ D_1(s_1 t_1))(p_i \circ D_2(s_2, t_2))}{(p_i \circ D_1(s_1 t_1)) + (p_i \circ D_2(s_2, t_2)) - (p_i \circ D_1(s_1 t_1))(p_i \circ D_2(s_2, t_2))}$ $\forall\ s_1 t_1 \in E_1$, $\forall\ s_2 t_2 \in E_2$.

Proposition 7.4.3. *Let G_1 and G_2 be the Dombi mPFGs of the graphs G_1^* and G_2^* respectively. The semi strong product $G_1 \bullet G_2$ is the Dombi mPFEG of $G_1^* \bullet G_2^*$.*

Proof. Consider $s \in V_1$, $s_2t_2 \in E_2$. Then

$$\begin{aligned}
& p_i \circ (D_1 \bullet D_2)((s, s_2)(s, y_2)) \\
= & \frac{(p_i \circ C_1(s))(p_i \circ D_2(s_2, t_2))}{(p_i \circ C_1(s)) + (p_i \circ D_2(s_2, t_2) - (p_i \circ C_1(s))(p_i \circ D_2(s_2, t_2)} \\
= & T(p_i \circ C_1(s), p_i \circ D_2(s_2, t_2)) \\
= & T(1, p_i \circ D_2(s_2, t_2)) \\
= & p_i \circ D_2(s_2, t_2) \\
\leq & \frac{(p_i \circ C_2(s_2))(p_i \circ C_2(s_2))}{(p_i \circ C_2(s_2)) + (p_i \circ C_2(s_2)) - (p_i \circ C_2(s_2))(p_i \circ C_2(s_2))} \\
= & \frac{(p_i \circ (C_1 \bullet C_2)(S))(p_i \circ (C_1 \bullet C_2)(T))}{(p_i \circ (C_1 \bullet C_2)(S)) + (p_i \circ (C_1 \bullet C_2)(T)) - (p_i \circ (C_1 \bullet C_2)(S))(p_i \circ (C_1 \bullet C_2)(T))} \\
& where,\ S = (s, s_2),\ T = (s, t_2)
\end{aligned}$$

Consider,

$$\begin{aligned}
& p_i \circ (D_1 \bullet D_2)((s_1, t_1)(s_2, t_2)) \\
= & \frac{(p_i \circ D_1(s_1, t_1))(p_i \circ D_2(s_2, t_2))}{(p_i \circ D_1(s_1, t_1)) + (p_i \circ D_2(s_2, t_2)) - (p_i \circ D_1(s_1, t_1))(p_i \circ D_2(s_2, t_2))} \\
= & T(p_i \circ D_1(s_1, t_1), p_i \circ D_2(s_2, t_2)) \\
\leq & T(\frac{(p_i \circ C_1(s_1))(p_i \circ C_1(t_1))}{(p_i \circ C_1(s_1)) + (p_i \circ C_1(t_1)) - (p_i \circ C_1(s_1))(p_i \circ C_1(t_1))}, \\
& \frac{(p_i \circ C_2(s_2))(p_i \circ C_2(t_2))}{(p_i \circ C_2(s_2)) + (p_i \circ C_2(t_2)) - (p_i \circ C_2(s_2))(p_i \circ C_2(t_2))})
\end{aligned}$$

Putting $k = p_i \circ C_1(s_1)$, $l = p_i \circ C_1(t_1)$, $m = p_i \circ C_2(s_2)$, $n = p_i \circ C_2(t_2)$

$$\begin{aligned}
& p_i \circ (D_1 \bullet D_2)((s_1, t_1)(s_2, t_2)) \\
\leq & T(\frac{kl}{k + l - kl}, \frac{mn}{m + n - mn}) \\
= & \frac{\frac{klmn}{(k+l-kl)(m+n-mn)}}{\frac{kl}{k+l-kl} + \frac{mn}{m+n-mn} - \frac{klmn}{(k+l-kl)(m+n-mn)}} \\
= & \frac{\frac{klmn}{(k+m-km)(l+n-ln)}}{\frac{km}{k+m-km} + \frac{ln}{l+n-ln} - \frac{klmn}{(k+m-km)(l+n-ln)}} \\
= & \frac{(p_i \circ (C_1 \bullet C_2)(S))(p_i \circ (C_1 \bullet C_2)(T))}{(p_i \circ (C_1 \bullet C_2)(S)) + (p_i \circ (C_1 \bullet C_2)(T)) - (p_i \circ (C_1 \bullet C_2)(S))(p_i \circ (C_1 \bullet C_2)(T))} \\
& where,\ S = (s_1, s_2),\ T = (t_1, t_2)
\end{aligned}$$

□

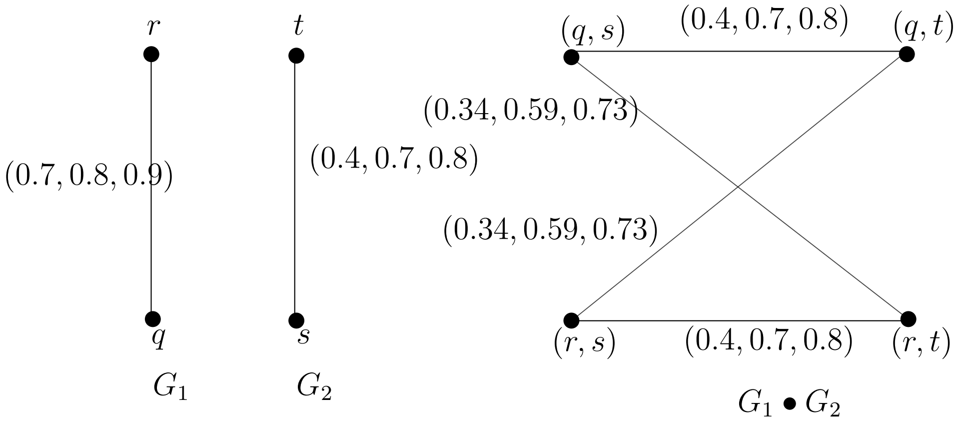

Figure 7.5: Semi strong product of two Dombi 3PFEG G_1 and G_2.

Example 7.4.4. *Here two Dombi 3PFEGs G_1 and G_2 with $C_1 = \{\frac{(1,1,1)}{q}, \frac{(1,1,1)}{r}\}$, $D_1 = \{\frac{(0.7,0.8,0.9)}{qr}\}$ and $C_2 = \{\frac{(1,1,1)}{s}, \frac{(1,1,1)}{t}\}$, $D_2 = \{\frac{(0.4,0.7,0.8)}{st}\}$ Then we have,*

$(D_1 \bullet D_2)((q,s),(q,t)) = (0.4, 0.7, 0.8)$

$(D_1 \bullet D_2)((r,s),(r,t)) = (0.4, 0.7, 0.8)$

$(B_1 \bullet B_2)((q,s),(r,t)) = (0.34, 0.59, 0.73)$

$(B_1 \bullet B_2)((q,t),(r,s)) = (0.34, 0.59, 0.73)$

Definition 7.4.5. *The strong product $G_1 \boxplus G_2 = (C_1 \boxplus C_2, D_1 \boxplus D_2)$ of the Dombi mPFGs $G_1 = (V_1, C_1, D_1)$ and $G_2 = (V_2, C_2, D_2)$ of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively, as follows $\forall\ i$,*

i) $p_i \circ (C_1 \boxplus C_2)(s_1, s_2) = \frac{(p_i \circ C_1(s_1))(p_i \circ C_2(s_2))}{(p_i \circ C_1(s_1)) + (p_i \circ C_2(s_2)) - (p_i \circ C_1(s_1))(p_i \circ C_2(s_2))}$, $\forall\ (s_1, s_2) \in V_1 \times V_2$.

ii) $p_i \circ (D_1 \boxplus D_2)((s, s_2)(s, t_2)) = \frac{(p_i \circ C_1(s))(p_i \circ D_2(s_2 t_2))}{(p_i \circ C_1(s)) + (p_i \circ D_2(s_2 t_2)) - (p_i \circ C_1(s_1))(p_i \circ D_2(s_2 t_2))}$ $\forall\ s \in V_1$, $\forall\ s_2 t_2 \in E_2$.

iii) $p_i \circ (D_1 \boxplus D_2)((s_1, t)(t_1, t)) = \frac{(p_i \circ D_1(s_1 t_1))(p_i \circ C_2(t))}{(p_i \circ D_1(s_1 t_1)) + (p_i \circ C_2(t)) - (p_i \circ D_1(s_1 t_1))(p_i \circ C_2(t))}$ $\forall\ s_1 t_1 \in E_1$ *and* $\forall\ t \in V_2$.

iv) $p_i \circ (D_1 \boxplus D_2)((s_1, x_2)(y_1, y_2)) = \frac{(p_i \circ D_1(s_1 t_1))(p_i \circ D_2(s_2, t_2))}{(p_i \circ D_1(s_1 t_1)) + (p_i \circ D_2(s_2, t_2)) - (p_i \circ D_1(s_1 t_1))(p_i \circ D_2(s_2, t_2))}$ $\forall\ s_1 t_1 \in E_1$, $\forall\ s_2 t_2 \in E_2$.

Proposition 7.4.4. *The strong product $G_1 \boxplus G_2$ of G_2 and G_1 is the domain mPFEG of $G_1^* \boxplus G_2^*$, where G_2 and G_1 be the Dombi mPFGs of the graphs G_2^* and G_1^*.*

Proof. This proposition is proof from the using of proposition 3.9 and 3.12. □

Example 7.4.5. *Consider two Dombi mPFG $G_1 = (V_1, C_1, D_1)$ and $G_2 = (V_2, C_2, D_2)$ where $C_1(s) = (1, 1, 1)\ \forall\ s \in V_1$ and $D_1 = \{\frac{(0.6,0.7,0.8)}{qr}\}$, $C_2(s) = (1, 1, 1)\ \forall\ s \in V_2$ and $D_2 = \{\frac{(0.3,0.5,0.8)}{uv}\}$ Then,* $(D_1 \boxplus D_2)((q, u)(q, v)) = (0.3, 0.5, 0.8)$

$(D_1 \boxplus D_2)((r, u)(r, v)) = (0.3, 0.5, 0.8)$

$(D_1 \boxplus D_2)((q, u)(r, u)) = (0.6, 0.7, 0.8)$

$(D_1 \boxplus D_2)((q, v)(r, v)) = (0.6, 0.7, 0.8)$

$(D_1 \boxplus D_2)((q, u)(r, v)) = (0.42, 0.41, 0.67)$

$(D_1 \boxplus D_2)((q, v)(r, u)) = (0.42, 0.41, 0.67)$.

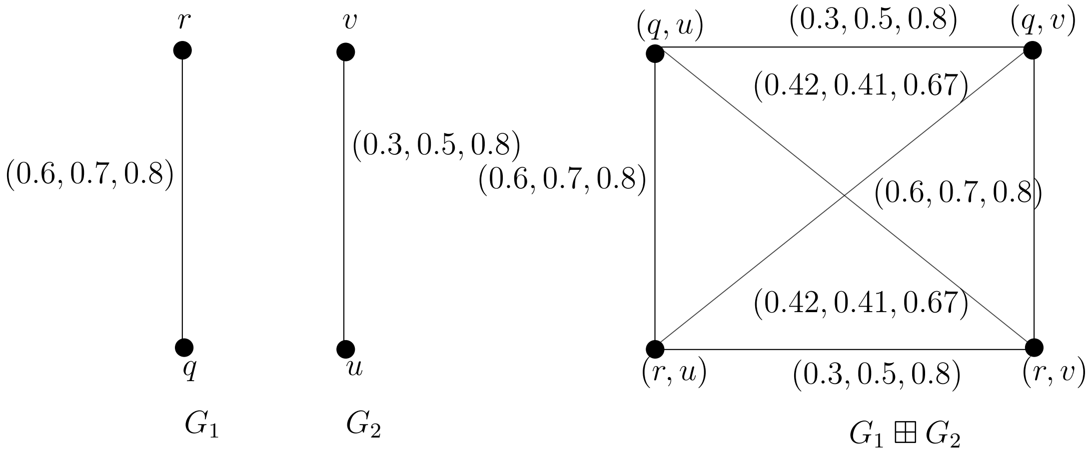

Figure 7.6: Strong product of two Dombi 3PFEG G_1 and G_2.

Definition 7.4.6. *The lexicographic product $G_1[G_2] = (C_1 \circ C_2, D_1 \circ D_2)$ of two Dombi mPFGs $G_1 = (V_1, C_1, D_1)$ and $G_2 = (V_2, C_2, D_2)$ is defined as, $\forall\ i$,*

i) $p_i \circ (C_1 \circ C_2)(s_1, s_2) = \frac{(p_i \circ C_1(s_1))(p_i \circ C_2(s_2))}{(p_i \circ C_1(s_1)) + (p_i \circ C_2(s_2)) - (p_i \circ C_1(s_1))(p_i \circ C_2(s_2))}$, $\forall\ (s_1, s_2) \in V_1 \times V_2$.

ii) $p_i \circ (D_1 \circ D_2)((s, s_2)(s, t_2)) = \frac{(p_i \circ C_1(s))(p_i \circ D_2(s_2 t_2))}{(p_i \circ C_1(s)) + (p_i \circ D_2(s_2 t_2)) - (p_i \circ C_1(s_1))(p_i \circ D_2(s_2 t_2))}$ $\forall\ s \in V_1$, $\forall\ s_2 t_2 \in E_2$.

iii) $p_i \circ (D_1 \circ D_2)((s_1, t)(t_1, t)) = \frac{(p_i \circ D_1(s_1 t_1))(p_i \circ C_2(t))}{(p_i \circ D_1(s_1 t_1)) + (p_i \circ C_2(t)) - (p_i \circ D_1(s_1 t_1))(p_i \circ C_2(t))}$ $\forall\ s_1 t_1 \in E_1$ *and* $\forall\ t \in V_2$.

iv) $p_i \circ (D_1 \circ D_2)((s_1, t_1)(s_1, t_2))$

$$= \frac{(p_i \circ C_2(s_2))(p_i \circ C_2(t_2))(p_i \circ D_1(s_1, t_1))}{(p_i \circ C_2(s_2))(p_i \circ C_2(t_2)) + (p_i \circ C_2(t_2))(p_i \circ D_(s_1, t_1)) + (p_i \circ C_2(s_2))(p_i \circ D_1(s_1, t_1)) - 2(p_i \circ C_2(s_2))(p_i \circ C_2(t_2))(p_i \circ D_1(s_1, t_1))}$$ $\forall\ s_1 t_1 \in E_1,\ s_2 \neq t_2$.

Proposition 7.4.5. *The lexicographic product $G_1[G_2]$ of two Dombi mPFG of G_1^* and G_2^* is the Dombi mPFEG of $G_1^*[G_2^*]$.*

Proof. Using the proposition 3.12, we get $\forall\ i$.

$p_i \circ (D_1 \circ D_2)((s, s_2)(s, t_2)) = \frac{(p_i \circ C_1(s))(p_i \circ D_2(s_2t_2))}{(p_i \circ C_1(s))+(p_i \circ D_2(s_2t_2))-(p_i \circ C_1(s_1))(p_i \circ D_2(s_2t_2))}\ \forall\ s \in V_1$, $\forall\ s_2t_2 \in E_2$.

And

$p_i \circ (D_1 \circ D_2)((s, s_2)(s, t_2)) = \frac{(p_i \circ C_1(s))(p_i \circ D_2(s_2t_2))}{(p_i \circ C_1(s))+(p_i \circ D_2(s_2t_2))-(p_i \circ C_1(s_1))(p_i \circ D_2(s_2t_2))}\ \forall\ s \in V_1$, $\forall\ s_2t_2 \in E_2$.

Now for $s_1t_1 \in E_1,\ s_2 \neq t_2$. Then

$p_i \circ (D_1 \circ D_2)((s_1, s_2)(t_1, t_2))$

$(let,\ p_i \circ C_2(s_2) = S,\ p_i \circ C_2(t_2) = T)$

$= \frac{(S)(T)(p_i \circ D_1(s_1,t_1))}{(S)(T)+(T)(p_i \circ D_(s_1,t_1))+(S)(p_i \circ D_1(s_1,t_1))-2(S)(T)(p_i \circ D_1(s_1,t_1))}$

$= T(T((p_i \circ C_1(s_1)), (p_i \circ C_1(t_2))), (p_i \circ D_1(s_1, t_1)))$

$= T(T(1, 1), (p_i \circ D_1(s_1, t_1)))$

$= (p_i \circ D_1(s_1, t_1))$

$\leq \frac{(p_i \circ C_1(s_1))(p_i \circ C_(t_1))}{(p_i \circ C_1(s_1))+(p_i \circ C_(t_1))-(p_i \circ C_1(s_1))(p_i \circ C_(t_1))}$

$= \frac{(p_i \circ (C_1 \circ C_2)(s_1,s_2))(p_i \circ (C_1 \circ C_2)(t_1,t_2))}{(p_i \circ (C_1 \circ C_2)(s_1,s_2))+(p_i \circ (C_1 \circ C_2)(t_1,t_2))-(p_i \circ (C_1 \circ C_2)(s_1,s_2))(p_i \circ (C_1 \circ C_2)(t_1,t_2))}$

Hence proved. □

Example 7.4.6. *Consider two Dombi mPFG $G_1 = (V_1, C_1, D_1)$ and $G_2 = (V_2, C_2, D_2)$ where $C_1(x) = (1, 1, 1)\ \forall\ x \in V_1$ and $D_1 = \{\frac{(0.2,0.4,0.7)}{qr}\}$, $C_2(x) = (1, 1, 1)\ \forall\ x \in V_2$, $D_2 = \{\frac{(0.5,0.6,0.9)}{st}, \frac{(0.3,0.5,0.7)}{tu}\}$. Then*

$(D_1 \circ D_2)((q, s)(r, s)) = (0.2, 0.4, 0.7),\ (D_1 \circ D_2)((a, d)(b, d)) = (0.2, 0.4, 0.7),$

$(D_1 \circ D_2)((q, u)(r, u)) = (0.2, 0.4, 0.7),\ (D_1 \circ D_2)((a, c)(a, d)) = (0.5, 0.6, 0.9),$

$(D_1 \circ D_2)((q, t)(q, u)) = (0.3, 0.5, 0.7),\ (D_1 \circ D_2)((b, c)(b, d)) = (0.3, 0.5, 0.7),$

$(D_1 \circ D_2)((r, t)(r, u)) = (0.3, 0.5, 0.7),\ (D_1 \circ D_2)((a, c)(b, d)) = (0.2, 0.4, 0.7),$

$(D_1 \circ D_2)((r, s)(q, t)) = (0.2, 0.4, 0.7),\ (D_1 \circ D_2)((a, d)(b, e)) = (0.2, 0.4, 0.7),$

$(D_1 \circ D_2)((q, u)(r, t)) = (0.2, 0.4, 0.7),\ (D_1 \circ D_2)((a, c)(b, e)) = (0.2, 0.4, 0.7),$

$(D_1 \circ D_2)((q, u)(r, s)) = (0.2, 0.4, 0.7).$

Definition 7.4.7. *Let C_1 be a mPF subset of V_i and D_i be a mPF subset of E_i, for $i = 1, 2$. Define the union $G_1 \cup G_2 = (C_1 \cup C_2, D_1 \cup D_2)$ of the Dombi mPFGs $G_1 = (V_1, C_1, D_1)$ and $G_2 = (V_2, C_2, D_2)$ as follows:*

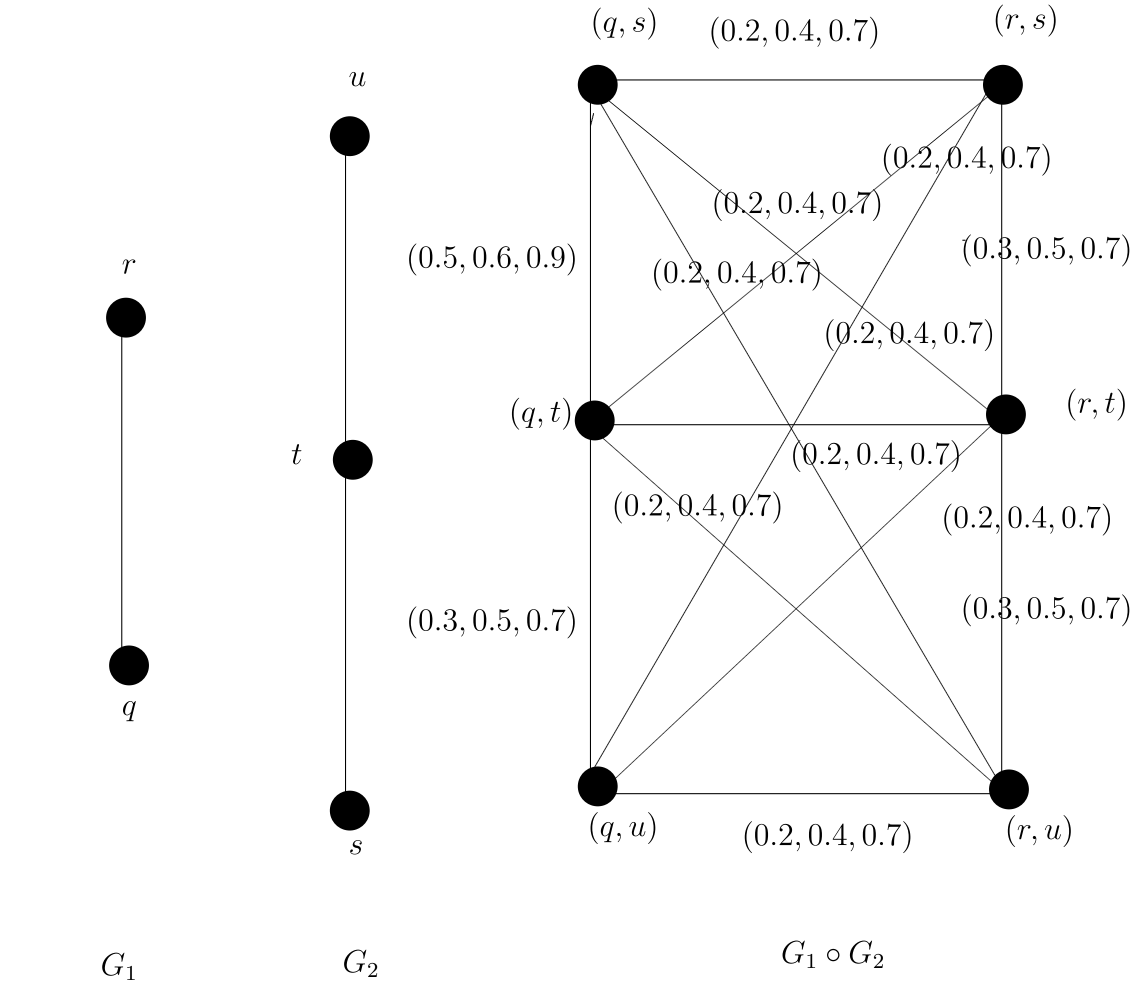

Figure 7.7: Lexicographic product of two Dombi 3PFEG G_1 and G_2.

$$p_i \circ (C_1 \cup C_2)(s) = \begin{cases} p_i \circ C_1(s), & if\ s \in V_1 \setminus V_2 \\ p_i \circ C_2(s), & if\ s \in V_2 \setminus V_1 \\ \frac{(p_i \circ C_1(s)) + (p_i \circ C_2(s)) - 2(p_i \circ C_1(s))(p_i \circ C_2(s))}{1-(p_i \circ C_1(s))(p_i \circ C_2(s))}, & if\ s \in V_1 \cap V_2 \end{cases}$$

$$p_i \circ (D_1 \cup D_2)(st) = \begin{cases} p_i \circ D_1(st), & if\ st \in E_1 \setminus E_2 \\ p_i \circ D_2(st), & if\ st \in E_2 \setminus E_1 \\ \frac{(p_i \circ D_1(st)) + (p_i \circ D_2(st)) - 2(p_i \circ D_1(st))(p_i \circ D_2(st))}{1-(p_i \circ D_1(st))(p_i \circ D_2(st))}, & if\ st \in E_1 \cap E_2 \end{cases}$$

Example 7.4.7. *We consider two Dombi mPFGs G_1 and G_2, where*
$C_1 = \{\frac{(0.5,0.6,0.8)}{q}, \frac{(0.8,0.9,0.7)}{r}, \frac{(0.5,0.6,0.7)}{c}\}$, $D_1 = \{\frac{(0.4,0.5,0.5)}{qr}, \frac{(0.4,0.5,0.5)}{rs}, \frac{(0.3,0.4,0.5)}{sq}\}$ *and*
$C_2 = \{\frac{(0.2,0.4,0.6)}{q}, \frac{(0.7,0.8,0.9)}{r}, \frac{(0.6,0.7,0.8)}{t}\}$, $D_2 = \{\frac{(0.1,0.3,0.5)}{qr}, \frac{(0.4,0.5,0.7)}{rt}, \frac{(0.1,0.3,0.5)}{qt}\}$
Then we have, $C_1 \cup C_2 = \{\frac{(0.5,0.6,0.7)}{s}, \frac{0.6,0.7,0.8}{t}, \frac{(0.55,0.68,0.84)}{q}, \frac{(0.86,0.92,0.92)}{r}\}$
$D_1 \cup D_2 = \{\frac{(0.43,0.58,0.75)}{qr}, \frac{0.4,0.5,0.5}{rs}, \frac{(0.4,0.5,0.7)}{rt}, \frac{(0.1,0.3,0.5)}{qt}, \frac{(0.3,0.4,0.5)}{sq}\}$.

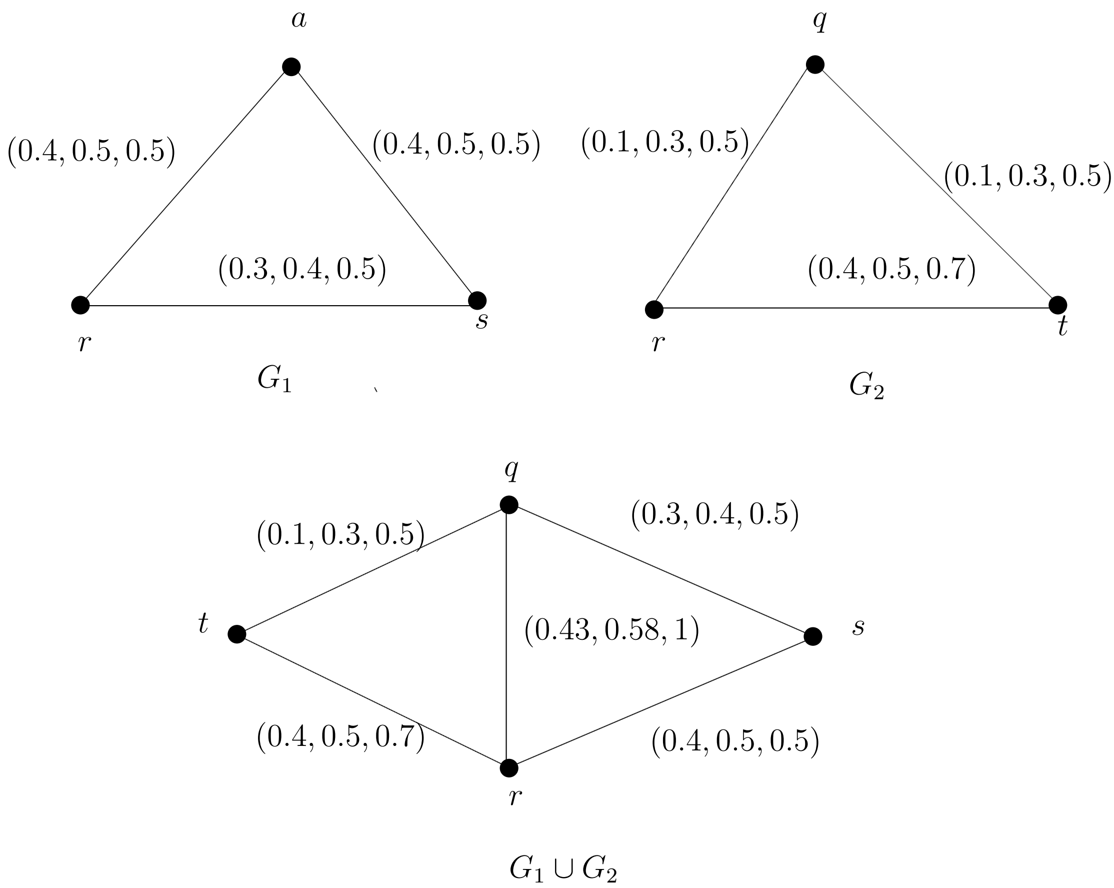

Figure 7.8: Union of two Dombi 3PFG G_1 and G_2.

Theorem 7.4.1. *Let $G_2 = (V_2, C_2, D_2)$ and $G_1 = (V_1, C_1, D_1)$ be two Dombi mPFG of the graphs $G_2^* = (V_2, E_2)$ and $G_1^* = (V_1, E_1)$, where $V_1 \cap V_2 = \phi$. Then the union $G_1 \cup G_2$ of G_1 and G_2 is the Dombi mPFG of $G_1 \cup G_2$.*

Proof. Suppose G_2 and G_1 are the Dombi mPFG of the graphs G_2^* and G_1^* respectively. Consider, $st \in E_1 \setminus E_2$. Then $\forall\ i$,

$p_i \circ (D_1 \cup D_2)(st)$

$= p_i \circ D_1(st)$

$\leq \frac{(p_i \circ C_1(s))(p_i \circ C_1(t))}{(p_i \circ C_1(s)) + (p_i \circ C_1(t)) - (p_i \circ C_1(s))(p_i \circ C_1(t))}$

$= \frac{(p_i \circ (C_1 \cup C_2)(s))(p_i \circ (C_1 \cup C_2)(t))}{(p_i \circ (C_1 \cup C_2)(s)) + (p_i \circ (C_1 \cup C_2)(t)) - (p_i \circ C_1(s))(p_i \circ (C_1 \cup C_2)(t))}$ as $V_1 \cap V_2 = \phi$

Similarly, if $st \in E_2 \backslash E_1$

Then $\forall\ i$, $p_i \circ (D_1 \cup D_2)(st) \leq \frac{(p_i \circ (C_1 \cup C_2)(s))(p_i \circ (C_1 \cup C_2)(t))}{(p_i \circ (C_1 \cup C_2)(s)) + (p_i \circ (C_1 \cup C_2)(t)) - (p_i \circ C_1(s))(p_i \circ (C_1 \cup C_2)(t))}$

And as $V_1 \cap V_2 = \phi$ so $E_2 \cap E_1 = \phi$.

Hence $G_1 \cup G_2$ is the Dombi mPFG of G_1^* and G_2^*. □

Definition 7.4.8. *The ring sum $G_1 \oplus G_2 = (C_1 \oplus C_2, D_1 \oplus D_2)$ of the Dombi mPFGs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is defined as for all i,*

$p_i \circ (C_1 \oplus C_2)(s) = p_i \circ (C_1 \cup C_2)(s)$ *if* $s \in V_1 \cup V_2$

$$p_i \circ (D_1 \oplus D_2)(st) = \begin{cases} p_i \circ D_1(st), & if\ st \in E_1 \setminus E_2 \\ p_i \circ D_2(st), & if\ st \in E_2 \setminus E_1 \\ 0, & if\ st \in E_1 \cap E_2 \end{cases}$$

Theorem 7.4.2. *The ring sum $G_1 \oplus G_2$ of two Dombi mPFGs G_1 and G_2 of G_1^* and G_2^* is the Dombi mPFG of $G_1^* \oplus G_2^*$.*

Proof. At first, we consider $st \in E_1 \setminus E_2$. Then there aries three possibilities

$i)s, t \in V_1 \setminus V_2$

$ii)s \in V_1 \setminus V_2, y \in V_1 \cap V_2$

$iii)s, t \in V_1 \cap V_2$.

i) Suppose $s, t \in V_1 \setminus V_2$. Then $\forall\ i$,

$$\begin{aligned} & p_i \circ (D_1 \oplus D_2)(st) \\ & = p_i \circ D_1(st) \\ & \leq \frac{(p_i \circ C_1(s))(p_i \circ C_1(t))}{(p_i \circ C_1(s)) + (p_i \circ C_1(t)) - (p_i \circ C_1(s))(p_i \circ C_1(t))} \\ & = T(p_i \circ C_1(s), p_i \circ C_1(t)) \\ & = T(p_i \circ (C_1 \oplus C_2)(s), p_i \circ (C_1 \oplus C_2)(t)) \\ & = \frac{(p_i \circ (C_1 \oplus C_2)(s))(p_i \circ (C_1 \oplus C_2)(t))}{(p_i \circ (C_1 \oplus C_2)(s)) + (p_i \circ (C_1 \oplus C_2)(t)) - (p_i \circ (C_1 \oplus C_2)(s))(p_i \circ (C_1 \oplus C_2)(t))} \end{aligned}$$

ii) Let $s, t \in V_1 \setminus V_2$, $t \in V_1 \cap V_2$. Then, $p_i \circ (D_1 \oplus D_2)(st)$

$$\begin{aligned} & = p_i \circ D_1(st) \\ & \leq \frac{(p_i \circ C_1(s))(p_i \circ C_1(t))}{(p_i \circ C_1(s)) + (p_i \circ C_1(t)) - (p_i \circ C_1(s))(p_i \circ C_1(t))} \\ & = T(p_i \circ C_1(s), p_i \circ C_1(t)) \\ & = T(p_i \circ (C_1 \oplus C_2)(s), p_i \circ C_1(t)) \qquad (i) \end{aligned}$$

Now we know, for all i,

$(p_i \circ C_1(t) - 1)^2 \geq 0$

$or,\ (p_i \circ C_1(t))^2 - 2(p_i \circ C_1(t)) + 1 \geq 0$

$or,\ 1 - 2(p_i \circ C_1(t)) \geq -(p_i \circ C_1(t))^2$

$or,\ \{1 - 2(p_i \circ C_1(t)\}(p_i \circ C_2(t)) \geq -(p_i \circ C_1(t))^2(p_i \circ C_2(t))$

$or,\ (p_i \circ C_1(t)) + (p_i \circ C_2(t)) - 2(p_i \circ C_1(t))(p_i \circ C_2(t)) \geq (p_i \circ C_1(t))$

$- (p_i \circ C_1(t))^2 (p_i \circ C_2(t))$

$$or, \ \frac{(p_i \circ C_1(t)) + (p_i \circ C_2(t)) - 2(p_i \circ C_1(t))(p_i \circ C_2(t))}{1-(p_i \circ C_1(t))(p_i \circ C_2(t))} \geq (p_i \circ C_1(t)) \qquad (ii)$$

Then from (i) and (ii) we get,

$$\begin{aligned}
& p_i \circ (D_1 \oplus D_2)(st) \\
\geq \ & T(p_i \circ (C_1 \oplus C_2)(s), p_i \circ C_1(t)) \\
\leq \ & T(p_i \circ (C_1 \oplus C_2)(s), \frac{(p_i \circ C_1(t)) + (p_i \circ C_2(t)) - 2(p_i \circ C_1(t))(p_i \circ C_2(t))}{1-(p_i \circ C_1(t))(p_i \circ C_2(t))}) \\
= \ & T(p_i \circ (C_1 \oplus C_2)(s), p_i \circ (C_1 \oplus C_2)(t))
\end{aligned}$$

iii) Suppose $s, t \in V_1 \cap V_2$. Then we get,

$$\begin{aligned}
p_i \circ (D_1 \oplus D_2)(st) &= p_i \circ D_1(st) \\
&\leq T(p_i \circ C_1(s), p_i \circ C_1(t))
\end{aligned}$$

Similarly, as from (ii)

$$\begin{aligned}
(p_i \circ C_1(s)) &\leq \frac{(p_i \circ C_1(s)) + (p_i \circ C_2(s)) - 2(p_i \circ C_1(s))(p_i \circ C_2(s))}{1-(p_i \circ C_1(s))(p_i \circ C_2(s))} \\
&= p_i \circ (C_1 \oplus C_2)(s)
\end{aligned}$$

$$\begin{aligned}
(p_i \circ C_1(t)) &\leq \frac{(p_i \circ C_1(t)) + (p_i \circ C_2(t)) - 2(p_i \circ C_1(t))(p_i \circ C_2(t))}{1-(p_i \circ C_1(t))(p_i \circ C_2(t))} \\
&= p_i \circ (C_1 \oplus C_2)(t)
\end{aligned}$$

So,

$$\begin{aligned}
p_i \circ (D_1 \oplus D_2)(st) &\leq T(p_i \circ C_1(s), p_i \circ C_1(t)) \\
&\leq T(p_i \circ (C_1 \oplus C_2)(s), p_i \circ (C_1 \oplus C_2)(t))
\end{aligned}$$

Again by symmetry, for $st \in E_2 \setminus E_1$, in the three possible case:

$$\begin{aligned} & p_i \circ (D_1 \oplus D_2)(st) \\ \leq \ & T(p_i \circ C_1(s), p_i \circ C_1(t)) \\ \leq \ & T(p_i \circ (C_1 \oplus C_2)(s), p_i \circ (C_1 \oplus C_2)(t)) \\ & \quad (let,\ (C_1 \oplus C_2)(s) = S,\ (C_1 \oplus C_2)(t) = T) \\ = \ & \frac{(p_i \circ S) + (p_i \circ T)}{(p_i \circ S) + (p_i \circ T) - (p_i \circ S)(p_i \circ T)} \end{aligned}$$

Hence proved.

□

Definition 7.4.9. *Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two Dombi mPFG. A homomorphism between G_1 and G_2 is a mapping $\phi : G_1 \to G_2$ satisfies the condition, $\forall\ i$,*

1) $p_i \circ C_1(s) \leq p_i \circ C_2(\phi(s))\ \forall\ s \in V_1$.

2) $p_i \circ D_1(st) \leq p_i \circ D_2(\phi(s), \phi(t))\ \forall\ st \in V_1$

Definition 7.4.10. *Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two Dombi mPFG. Then an isomorphism between G_1 and G_2 is a mapping $\phi : G_1 \to G_2$ satisfies the condition, $\forall\ i$*

1) $p_i \circ C_1(s) = p_i \circ C_2(\phi(s))\ \forall\ s \in V_1$.

2) $p_i \circ D_1(st) = p_i \circ D_2(\phi(s), \phi(t))\ \forall\ st \in E_1$.

Definition 7.4.11. *A week isomorphism $\phi : G_1 \to G_2$ is a bijective mapping between G_1 and G_2 which satisfies*

1) $p_i \circ C_1(s) = p_i \circ C_2(\phi(s))\ \forall\ s \in V_1$.

Definition 7.4.12. *A co-week isomorphism $\phi : G_1 \to G_2$ is a bijective mapping between G_1 and G_2 which satisfies,*

1) $\forall\ i,\ p_i \circ D_1(st) = p_i \circ D_2(\phi(s), \phi(t))\ \forall\ st \in V_1$.

Definition 7.4.13. *A Dombi mPFG $G = (C, D)$ is called self-complementary if $G = (C, D) \cong \bar{G} = (\bar{C}, \bar{D})$.*

Definition 7.4.14. *Let $G = (C, D)$ be a Dombi mPFG of the graph $G^* = (V, E)$. Then the complement of G is a Dombi mPFG $\bar{G} = (\bar{C}, \bar{D})$ where $\bar{C}(s) = C(s)\ \forall\ s \in V$ and $\bar{D}$ is defined as $\forall\ i$,*

$$p_i \circ \bar{D}(s,t) = \begin{cases} \frac{(p_i \circ C(s))(p_i \circ C(t))}{(p_i \circ C(s)) + (p_i \circ C(t)) - (p_i \circ C(s))(p_i \circ C(t))}, & if\ p_i \circ D(s,t) = 0, \\ \frac{(p_i \circ C(s))(p_i \circ C(t))}{(p_i \circ C(s)) + (p_i \circ C(t)) - (p_i \circ C(s))(p_i \circ C(t))} - p_i \circ D(s,t), & if\ 0 \le p_i \circ D(s,t) \le 1. \end{cases}$$

Example 7.4.8. *Consider a Dombi mPFG $G = (C, D)$ of a graph $G^* = (V, E)$ where $V = \{q, r, s, t\}$ and $E = \{qr, rs, qt\}$. Then $C = \{\frac{(0.3,0.5,0.6)}{a}, \frac{(0.2,0.3,0.5)}{b}, \frac{(0.6,0.7,0.9)}{c}, \frac{(0.2,0.4,0.6)}{d}\}$ and $D = \{\frac{(0.14,0.23,0.38)}{qr}, \frac{(0.17,0.26,0.47)}{rs}, \frac{(0.14,0.28,0.42)}{qt}\}$. Now the complement Dombi mPFG is $\bar{G} = (\bar{C}, \bar{D})$ where $\bar{C}(s) = C(s)\ \forall\ s \in V$ and $\bar{D} = \{\frac{(0.17,0.34,0.56)}{st}, \frac{(0.25,0.41,0.56)}{qs}, \frac{(0.11,0.2,0.37)}{rt}\}$.*

Proposition 7.4.6. *Let $G = (C, D)$ be a self complementary Dombi mPFG, then $\sum\limits_{s \ne t} p_i \circ D(s,t) = \frac{1}{2} \sum\limits_{s \ne t} \frac{(p_i \circ C(s))(p_i \circ C(t))}{(p_i \circ C(s)) + (p_i \circ C(t)) - (p_i \circ C(s))(p_i \circ C(t))}$.*

Proof. Let G be a self-complementary Dombi mPFG. So there is an bijective mapping $\phi : V \to V$ s.t. $\forall\ i$

1) $p_i \circ C(s) = p_i \circ \bar{C}(\phi(s))\ \forall\ s \in V_1$.

2) $p_i \circ D(st) = p_i \circ \bar{D}(\phi(s), \phi(t))\ \forall\ st \in E_1$.

Let, $(p_i \circ \bar{C}(\phi(s)) = S,\ (p_i \circ \bar{C}(\phi(t)) = T$. Then, $\forall\ i$,

$$p_i \circ \bar{D}(\phi(s), \phi(t)) = \frac{S)(T)}{(S) + (T) - (p_i \circ \bar{C}(\phi(s)))(p_i \circ \bar{C}(\phi(t)))} - p_i \circ D(\phi(s), \phi(t))$$

$$or, \quad p_i \circ D(s,t) = \frac{(p_i \circ C(s))(p_i \circ C(t))}{(p_i \circ C(s)) + (p_i \circ C(t)) - (p_i \circ C(s))(p_i \circ C(t))} - p_i \circ D(\phi(s), \phi(t))$$

$$or, \quad \sum_{s \ne t} p_i \circ D(s,t) + \sum_{s \ne t} p_i \circ D(\phi(s), \phi(t)) = \sum_{s \ne t} \frac{(p_i \circ C(s))(p_i \circ C(t))}{(p_i \circ C(s)) + (p_i \circ C(t)) - (p_i \circ C(s))(p_i \circ C(t))}$$

$$or, \quad 2 \sum_{s \ne t} p_i \circ D(s,t) = \sum_{s \ne t} \frac{(p_i \circ C(s))(p_i \circ C(t))}{(p_i \circ C(s)) + (p_i \circ C(t)) - (p_i \circ C(s))(p_i \circ C(t))}$$

$$or, \quad \sum_{s \ne t} p_i \circ D(s,t) = \frac{1}{2} \sum_{s \ne t} \frac{(p_i \circ C(s))(p_i \circ C(t))}{(p_i \circ C(s)) + (p_i \circ C(t)) - (p_i \circ C(s))(p_i \circ C(t))}$$

□

Proposition 7.4.7. *The complements of two isomorphic Dombi mPFGs are isomorphic and conversely.*

Proof. Let G_1 and G_2 be two isomorphic Dombi mPFGs. So there is a bijection mapping $\phi : G_1 \to G_2$ s.t. $\forall\ i$,

1) $p_i \circ C_1(s) = p_i \circ C_2(\phi(s))\ \forall\ s \in V_1$.

2) $p_i \circ D_1(st) = p_i \circ D_2(\phi(s), \phi(t))\ \forall\ st \in E_1$.

Now we have,

$$\begin{aligned} p_i \circ \bar{D_1}(s,t) &= \frac{(p_i \circ C_1(s))(p_i \circ C_1(t))}{(p_i \circ C_1(s)) + (p_i \circ C_1(t)) - (p_i \circ C_1(s))(p_i \circ C_1(t))} - p_i \circ D_1(s,t) \\ &= \frac{(p_i \circ C_2(\phi(s)))(p_i \circ C_2(\phi(t)))}{(p_i \circ C_2(\phi(s))) + (p_i \circ C_(\phi(t))) - (p_i \circ C_2(\phi(s)))(p_i \circ C_2(\phi(t)))} \\ &\qquad - p_i \circ D_2(\phi(s), \phi(t)) \\ &= p_i \circ \bar{D_2}(\phi(s), \phi(t)) \end{aligned}$$

Hence, $\bar{G_1} \cong \bar{G_2}$.

Similarly, we can prove the converse. □

Proposition 7.4.8. *Let $G = (C, D)$ be a Dombi mPFG with $\forall\ i$, $p_i \circ D(s,t) = \frac{1}{2}\frac{(p_i \circ C(s))(p_i \circ C(t))}{(p_i \circ C(s)) + (p_i \circ C(t)) - (p_i \circ C(s))(p_i \circ C(t))}$ $\forall\ s, t \in V$. Then G is self complementary.*

Proof. Let $G = (C, D)$ be a Dombi mPFG with $\forall\ i$,
$p_i \circ D(s,t) = \frac{1}{2}\frac{(p_i \circ C(s))(p_i \circ C(t))}{(p_i \circ C(s)) + (p_i \circ C(t)) - (p_i \circ C(s))(p_i \circ C(t))}$ $\forall\ s, t \in V$.

Now we consider an identity mapping $I : V \to V$ which is an isomorphism from G to $\bar{G}$. Clearly $\forall\ i$,

$$\begin{aligned} p_i \circ \bar{C}(I(s)) &= p_i \circ \bar{C}(s) \\ &= p_i \circ C(s) \end{aligned}$$

Again,

$$\begin{aligned}
& p_i \circ \bar{D}(I(s), I(t)) \\
= \;& p_i \circ \bar{D}(s,t) \\
= \;& \frac{(p_i \circ C(s))(p_i \circ C(t))}{(p_i \circ C(s)) + (p_i \circ C(t)) - (p_i \circ C(s))(p_i \circ C(t))} - p_i \circ D(s,t) \\
= \;& \frac{(p_i \circ C(s))(p_i \circ C(t))}{(p_i \circ C(s)) + (p_i \circ C(t)) - (p_i \circ C(s))(p_i \circ C(t))} - \\
& \qquad \frac{1}{2}\frac{(p_i \circ C(s))(p_i \circ C(t))}{(p_i \circ C(s)) + (p_i \circ C(t)) - (p_i \circ C(s))(p_i \circ C(t))} \\
= \;& \frac{1}{2}\frac{(p_i \circ C(s))(p_i \circ C(t))}{(p_i \circ C(s)) + (p_i \circ C(t)) - (p_i \circ C(s))(p_i \circ C(t))} \\
= \;& p_i \circ D(s,t)
\end{aligned}$$

So, G is self-complementary. □

Proposition 7.4.9. *Let G_1 and G_2 be two weak isomorphic Dombi mPFGs. Then the complements of G_1 and G_2 are weak isomorphic.*

Proof. Let G_1 and G_2 be two weak isomorphic Dombi mPFGs. So there is a bijective mapping $\phi : V_1 \to V_2$ satisfying for all i,

1) $p_i \circ C_1(s) = p_i \circ C_2(\phi(s))$ for all $s \in V_1$.

2) $p_i \circ D_1(s,t) = p_i \circ D_2(\phi(s), \phi(t))$ for all $(s,t) \in E_1$

Now, $p_i \circ D_1(s,t) \leq p_i \circ D_2(\phi(s), \phi(t))$

or, $p_i \circ D_1(s,t) - \frac{(p_i \circ C_1(s))(p_i \circ C_1(t))}{(p_i \circ C_1(s)) + (p_i \circ C_1(t) - (p_i \circ C_1(s))(p_i \circ C_1(t)))}$

$$\leq p_i \circ D_2(\phi(s), \phi(t)) - \frac{(p_i \circ C_1(s))(p_i \circ C_1(t))}{(p_i \circ C_1(s)) + (p_i \circ C_1(t) - (p_i \circ C_1(s))(p_i \circ C_1(t)))}$$

or, $p_i \circ B_1(s,t) - \frac{(p_i \circ C_1(s))(p_i \circ C_1(t))}{(p_i \circ C_1(s)) + (p_i \circ C_1(t) - (p_i \circ C_1(s))(p_i \circ C_1(t)))}$

$$\leq p_i \circ D_2(\phi(s), \phi(t)) - \frac{(p_i \circ C_2(\phi(s)))(p_i \circ C_2(\phi(t)))}{(p_i \circ C_2(\phi(s))) + (p_i \circ C_2(\phi(t)) - (p_i \circ C_2(\phi(s)))(p_i \circ C_2(\phi(t)))}$$

or, $\frac{(p_i \circ C_1(s))(p_i \circ C_1(t))}{(p_i \circ C_1(s)) + (p_i \circ C_1(t) - (p_i \circ C_1(s))(p_i \circ C_1(t)))} - p_i \circ D_1(s,t)$

$$\geq \frac{(p_i \circ C_2(\phi(s)))(p_i \circ C_2(\phi(t)))}{(p_i \circ C_2(\phi(s))) + (p_i \circ C_2(\phi(t)) - (p_i \circ C_2(\phi(s)))(p_i \circ C_2(\phi(t)))} - p_i \circ D_2(\phi(s), \phi(t))$$

or, $p_i \circ \bar{D}_1(s,t) \geq p_i \circ \bar{D}_2(\phi(s), \phi(t))$

or, $p_i \circ \bar{D}_2(\phi(s), \phi(t)) \leq p_i \circ \bar{D}_1(\phi^{-1}(\phi(s)), \phi^{-1}(\phi(t)))$

$or,\ p_i \circ \bar{D}_2(s_1, t_1) \leq p_i \circ \bar{D}_1(\phi^{-1}(s_1), \phi^{-1}(t_1))$for all $s_1, t_1 \in V_2$

again we know,

$p_i \circ C_1(s) = p_i \circ C_2(\phi(s))$

$or,\ p_i \circ C_2(\phi(s)) = p_i \circ C_1(\phi^{-1}(\phi(s)))$ for all $\phi(s) \in V_2$

$or,\ p_i \circ C_2(t_1) = p_i \circ C_1(\phi^{-1}(t_1))$ for all $t_1 \in V_2$

Hence $\bar{G}_1$ and $\bar{G}_2$ are weak isomorphic. □

7.5 Summary

The fresh Dombi mPFG idea is launched in this article. The ring sum, join and direct product of two Dombi mPFGs has been proven to be the Dombi mPFGs. In particular, however, the lexicographic product, the strong product, the semi-strong product and the Cartesian product of two Dombi mPFGs are not Dombi mPFGs. The Dombi mPFG can portray all types of networks' uncertainty well.