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The Novel Concept of Roman Domination in Fuzzy Graph

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ABSTRACT

Suppose $\mathfrak{G} = (X, E)$ is a graph. A Roman dominating function (RDF) of $\mathfrak{G}$ is a function $g: X \to \{0,1,2\}$ such that every vertex m for which $g(m) = 0$ has a neighbor s with $g(s) = 2$. The weight of an RDF g is $w(m) = \sum_{m\in X} g\,(m)$. The Roman domination number (RDN) of a graph $\mathfrak{G}$, denoted by $\wp_R(\mathfrak{G})$, is the minimum weight of all possible RDFs. In this study, we define RDF on fuzzy graph (FG). Our purpose is to develop a notion of the RDF and also to present some basic definitions, notations, remarks, and proofs related to RDF on FGs.

Keywords: Fuzzy graph, dominating set, Roman domination function.

AMS Mathematics Subject Classification (2020): 05C72

1. Introduction

Ian Stewart discussed the strategy of Emperor Constantine for defending the Roman Empire. Cockayne et al. (2004) introduced the notion of Roman domination in graph [1]. Akram [2] presented some results on the strong Roman domination number (RDN) of graphs. Roushini leely pushpam [3,4] defined new notions of Roman domination in graphs. Varieties of Roman domination II are introduced by Chellali [5].

Graphs, from ancient times to the present day, have played a very important role in various fields, including computer science and social networks, so with the help of the vertices and edges of a graph, the relationships between objects and elements in a social group can be easily introduced. But, some phenomena in our lives have a wide range of complexities that make it impossible for us to express certainty. These complexities and ambiguities were reduced with the introduction of FSs by Zadeh [6]. The subject of FGs was presented by Rosenfeld [7]. An FG has good capabilities in dealing with problems that cannot be explained by weight graphs. Somasundaram discussed in [8], the concept of domination and determines the domination number for several fuzzy graphs. Prassanna [9] studied domination in FGs. Ghaffari [10] discussed the Roman domination problem with uncertain positioning and deployment costs. The definition of a Roman dominating function is given implicitly in [11,12]. Roman domination in graphs has been studied, for example in $[13-19]$. Some results of FGs were introduced in $[20-33]$.

Suppose $\mathfrak{G}=(X,E)$ is a graph of order $|X|=n$. For any vertex $m\in X$, the open neighbourhood of m is the set $\mathfrak{N}(m)=\{s\in X|sm\in E\}$ and the closed neighbourhood of m is the set $\mathfrak{N}[m]=\mathfrak{N}(m)\cup\{m\}$. The degree of m, denoted by $deg(X)$, is the total number of neighbours of m. In other words $deg(m)=|\mathfrak{N}(m)|$. For a set $\mathfrak{S}\subseteq X$, the open neighbourhood is $\mathfrak{N}(\mathfrak{S})=\cup_{m\in\mathfrak{S}}$ and the closed neighbourhood is $\mathfrak{N}[\mathfrak{S}]=\mathfrak{N}(\mathfrak{S})\cup\mathfrak{S}$. A subset $\mathfrak{S}\subseteq X$ is a domination set(DS) of $\mathfrak{G}$, if, for any vertex $m\in X-\mathfrak{S}$, there exists a vertex $s\in\mathfrak{S}$ such that $ms\in E$. The DN of $\mathfrak{G}$, is the minimum cardinality of DS and is denoted by $\wp(\mathfrak{G})$. A DS of cardinality $\wp(\mathfrak{G})$ is called a $\wp-$set of $\mathfrak{G}$. A Roman dominating function (RDF) of FG $\mathfrak{G}$ is a function $g: X\rightarrow\{0,1,2\}$ such that every vertex m for which $g(m)=0$ has a neighbour s with $g(s)=2$. The weight of an RDF g is $w(m)=\sum_{m\in X} g\,(m)$.

A graph $\mathfrak{G}$ is bipartite if the vertex set can be partitioned into two disjoint subsets S_1 and S_2 such that the vertices in S_1 are only adjacent to vertices in S_2 and vice versa. The complete bipartite graph is denoted by $K_{r,s}$, where $|X|=S_1\cup S_2$, $|S_1|=r$,$|S_2|=s$, S_1 and S_2 are independent sets and every vertex in S_1 is adjacent to every vertex in S_2.

More than 50 types of domination parameters have been studied by different authors. In this paper, we developed the concept of RDF on FGs and also, presented a new definition of it.

2. Preliminaries

In this section, we present some preliminary results which will be used throughout the paper.

Definition 2.1. Suppose X is a finite non-empty set, and E is a collection of all two-element subsets of X. A graph is a pair $\mathfrak{G}^*=(X,E)$ where X and $E\subseteq X\times X$ are the set of vertices and the set of edges of $\mathfrak{G}^*$, respectively.

Definition 2.2. Assume $\mathfrak{G}^*=(X,E)$ is a graph. A subset $\mathfrak{D}$ of a vertex set $X(\mathfrak{G}^*)$ is DS of a graph $\mathfrak{G}^*$, if for every vertex $m\in X(\mathfrak{G}^*)-\mathfrak{D}$ there exists a vertex $n\in\mathfrak{D}$ such that mn is an edge of $\mathfrak{G}^*$. The domination number(DN) $\xi(\mathfrak{G}^*)$ of $\mathfrak{G}^*$ is the smallest cardinality of a DS $\mathfrak{D}$ of $\mathfrak{G}^*$.

Definition 2.3. An FG $\mathfrak{G}=(\phi,\psi)$ is a pair of function $\phi: X\rightarrow[0,1]$ and $\psi: X\times X\rightarrow[0,1]$ such that, for all $m,n\in X$,

$$\psi(m,n)\leq\min\{\phi(m),\phi(n)\},$$

Definition 2.4. The order p and size q of the FG $\mathfrak{G}=(\phi,\psi)$ are described by:

$$|p|=\sum_{m\in X}\phi\,(m)\quad,\quad|q|=\sum_{mn\in E}\psi\,(mn)$$

Suppose $\mathfrak{G}$ is an FG on X and $\mathfrak{S}\subseteq X$, then the cardinality of $\mathfrak{S}$ is defined as:

$$|\mathfrak{S}|=\sum_{m\in\mathfrak{S}}\phi\,(m)$$

Definition 2.5. A path $\mathfrak{P}$ of length l is a sequence of distinct vertices $x_0, x_1, x_2, \ldots, x_l$ such that $\psi(x_{k-1}, x_k) > 0, k = 1,2,3, \ldots, l$. The degree of membership of a weakest edge is defined as its strength. The strength of connectedness between two vertices m and n is defined as the maximum of the strength of all paths between m and n is denoted by $\psi^{\infty}(m,n)$ or $CONN_{\mathfrak{G}}(m,n)$.

The strength of the connectedness between two vertices m and n in an FG $\mathfrak{G}$ is

$$\psi^{\infty}(m,n) = \sup\{\psi^{l}(m,n) : l = 1,2,3, \ldots\}$$

where

$$\psi^{l}(m,n) = \sup\{\psi(m,x_1) \wedge \psi(x_1,x_2) \wedge \psi(x_2,x_3) \wedge \ldots \wedge \psi(x_{l-1},n)\}$$

Definition 2.6. An edge mn is called to be a strong edge (SE) if $\psi^{\infty}(m,n)$ if $\psi(m,n) = 0$ for each $n \in X$, then m is named an isolated vertex.

An edge of an FG is named strong if its weight is at least as great as the strength of the connectedness of its end vertices when it is deleted. Note that, $CONN_{\mathfrak{G}-mn}(m,n)$ is the strength of the connectedness between m and n in an FG obtained from $\mathfrak{G}$ by deleting the edge mn.

Definition 2.7. Assume mn is an edge in FG $\mathfrak{G}$ then,
An edge mn is $\alpha -$ strong if $CONN_{\mathfrak{G}-mn}(m,n) < \psi(m,n)$.
An edge mn is $\beta -$ strong if $CONN_{\mathfrak{G}-mn}(m,n) = \psi(m,n)$.
An edge mn is $\delta -$ strong if $CONN_{\mathfrak{G}-mn}(m,n) > \psi(m,n)$.

Therefore, an edge mn is an SE if it is either $\alpha -$strong or $\beta -$strong.

Definition 2.8. Two vertices m and n in an FG $\mathfrak{G}$ are called adjacent if $\psi(m,n) > 0$ and m and n are named neighbours. The collection of all neighbours of m is denoted by $N(m)$. An edge mn of an FG is named an effective edge if $\psi(m,n) = \phi(m) \wedge \phi(n)$. Thus, m and n are named effective neighbours (EN). The set of all EN of m is named EN of m and is shown by $\mathfrak{EN}(m)$. Also, n is named the strong neighbour (SN) of m if edge mn is strong. The set of all SNs of m is named the open SN of m and is denoted by $\mathfrak{N}_s(m)$. The close SN $\mathfrak{N}_s[m]$ is defined as $\mathfrak{N}_s[m] = \mathfrak{N}_s(m) \cup \{m\}$.

Definition 2.9. Suppose $\mathfrak{G}$ is an FG and $\mathfrak{D} \subseteq X$, $\mathfrak{D}$ is a DS if for each $m \in X - \mathfrak{D}$ there exist $n \in \mathfrak{D}$ such that,
$(ii)(m,n)$ is an SE.
$(ii)\phi(m) \leq \phi(n)$.

A DS of an FG minimum number of vertices is named, minimum DS. The minimum DS of an FG $\mathfrak{G}$ is named the DN of an FG and is denoted by $\wp(\mathfrak{G})$.

Definition 2.10. The weight of a strong domination set (SDS) $\mathfrak{D}$ is described as, $\mathfrak{W}(\mathfrak{D}) = \sum_{m \in \mathfrak{D}} \psi(m,n)$, such that $\psi(m,n)$ is the minimum weight(MW) of the SE incident on m. The strong DN of an FG $\mathfrak{G}$ is as the MW of SDSs of $\mathfrak{G}$ and is shown by $\wp_s(\mathfrak{G})$. A minimum SDS in an FG $\mathfrak{G}$ is an SDS of MW.

Definition 2.11. A Roman dominating function (RDF) on graph $\mathfrak{G}^* = (X, E)$ is defined as a function $g: X \to \{0,1,2\}$ satisfying the condition that every vertex m for which $g(m) = 0$ is adjacent to at least one vertex n for which $g(n) = 2$. The weight of an RDF is the value $g(X) = \sum_{m \in X} g\,(m)$. The Roman domination number (RDN) of a graph $\mathfrak{G}^*$, denoted by $\wp_R(\mathfrak{G}^*)$, is the MW of RDFs on $\mathfrak{G}^*$.

A graph $\mathfrak{G}^*$ is a Roman graph if $\wp_R(\mathfrak{G}^*) = 2\wp(\mathfrak{G}^*)$ Suppose (X_0, X_0, X_0) is the ordered partition of X induced by g, such that $X_k = \{m \in X | g(m) = k\}$ and $X_k = l_k$, for $k = 0, 1, 2$.

Note that there exists a 1-1 correspondence between the functions g and the ordered partition (X_0, X_0, X_0) of X. $\mathfrak{d}(m, n)$ is the distance between two vertices m and n in a graph $\mathfrak{G}^*$ that is defined as the number of edges in the shortest path connecting them.

Example 2.12. A graph $\mathfrak{G}^* = (X, E)$ with $X = \{z_1, z_2, z_3, z_4, z_5, z_6\}$ is shown in Fig 1. A function $g(z_i): X \to \{0,1,2\}$ is defined. Assume $g(z_1) = 0, g(z_2) = 2, g(z_3) = 0, g(z_4) = 1, g(z_5) = 2$.

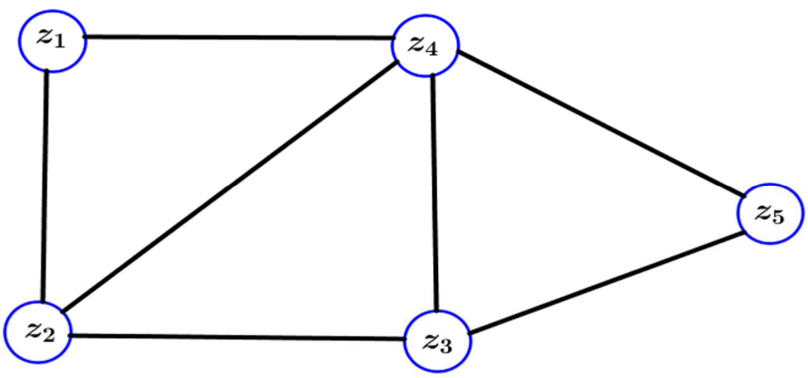

Figure 1: Roman graph

The $\mathfrak{D} = \{z_2, z_4\}$ is a DS of graph $\mathfrak{G}^*$. The DN of graph $\mathfrak{G}^*$ is $\wp(\mathfrak{G}^*) = 2$. The RDN is $\wp_R(\mathfrak{G}^*) = 2 \times 1 + 1 \times 1 = 3$.

In Table 1, we show the essential notations.

Some essential notations.

Table 1: Some essential notations.

Notation	Meaning
FS	Fuzzy Set
FG	Fuzzy Graph
DS	Dominating Set
DN	Domination Number
SE	Strong Edge
SDS	Strong Domination Set
MW	Minimum Weight

EN	Effective Neighbor
SN	Strong Neighbor
RDF	Roman Dominating Function
RDN	Roman Domination Number

3. Roman domination in fuzzy graphs

In this section, we defined a new notion of Roman domination in FGs.

Definition 3.1. Suppose $\mathfrak{G} = (\phi, \psi)$ is a FG on $\mathfrak{G}^* = (X, E)$. Assume a function $g: X \rightarrow \{0,1,2\}$ and let (X_0, X_1, X_2) is a ordered partition of X induced by g, where $X_l = \{m \in X | g(m) = l\}$ and $|X_l| = k_l$ for $l = 0,1,2$.

A RDF on FG is defined as a function g satisfying the condition:
there exists a strong edge between every vertex as m with $g(m) = 0$ and one vertex v with $g(v) = 2$. Let $\phi(m)$ be a membership degree of vertices, the weight of a RDF on FG $\mathfrak{G}$ is the value $g(X) = \sum_{m \in X} g\,(m)\phi(m)$. The RDN of an FG $\mathfrak{G}$, denoted by $\wp_R(\mathfrak{G})$, is the MW of RDFs on $\mathfrak{G}$.

An FG $\mathfrak{G}$ is a Roman FG if $\wp_R(\mathfrak{G}) \leq 2\wp(\mathfrak{G})$.

Example 3.2. Consider $\mathfrak{G}$ is an FG on $\mathfrak{G}^* = (X, E)$. Assume $X = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ and $E = \{c_1c_2, c_1c_4, c_2c_3, c_2c_4, c_3c_4, c_3c_5, c_4c_5, c_5c_6\}$ and $\mathfrak{D} \subseteq X$. The $\mathfrak{D} = \{c_1, c_3, c_6\}$ is a DS on FG in Fig 2. The DN of FG $\mathfrak{G}$ is $\wp(\mathfrak{G}) = 0.9$. The strong edges is shown in Table 2.

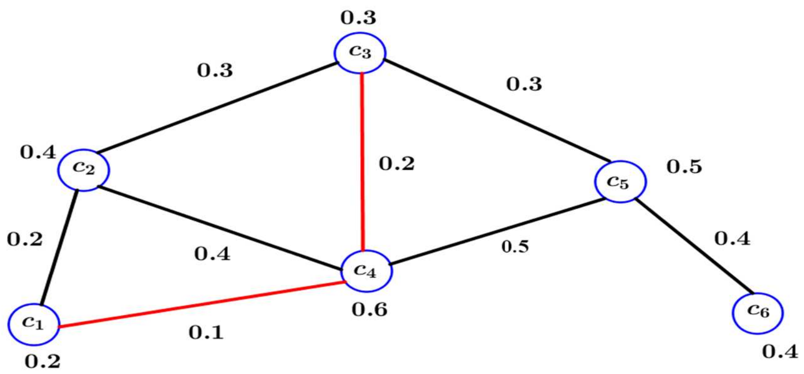

Figure 2: A fuzzy graph

Table 2. The strong edges

Strong edges	c_1c_2	c_2c_4	c_2c_3	c_3c_5	c_4c_5	c_5c_6
value	0.2	0.4	0.3	0.3	0.5	0.4

Assume a function $g: X \rightarrow \{0,1,2\}$, and we consider
$g(c_1) = 1, \quad g(c_2) = 0, \quad g(c_3) = 2, \quad g(c_4) = 2, \quad g(c_5) = 0, \quad g(c_6) = 1.$
Thus,

$$|g(X)| = \sum_{c_i \in X} g\,(c_i)\phi(c_i) = 1 \times 0.2 + 2 \times 0.3 + 2 \times 0.6 + 1 \times 0.4 = 2.4$$

The RDN of FG $\mathfrak{G}$ is $\wp_R(\mathfrak{G}) = 1 \times 0.2 + 2 \times 0.3 + 1 \times 0.4 = 1.2$.

Theorem 3.3. *For every FG $\mathfrak{G}$, we have $\wp(\mathfrak{G}) \leq \wp_R(\mathfrak{G}) \leq 2\wp(\mathfrak{G})$.*

Proof: Suppose $g = (X_0, X_1, X_2)$ is a $\wp_R$ function and $\mathfrak{D}$ is a $\wp -$ set of $\mathfrak{G}$. Then $X_1 \cup X_2$ is a DS of $\mathfrak{G}$ and $(X - \mathfrak{D}, \emptyset, \mathfrak{D})$ is an RDF. Therefore,

$$\wp(\mathfrak{G}) = \sum_{m \in \mathfrak{D}} \phi\,(m) \leq \sum_{m \in X_1 \cup X_2} \phi\,(m) g(m) = \wp_R(\mathfrak{G}).$$

But $\wp_R(\mathfrak{G}) \leq 2|\mathfrak{D}| = 2\sum_{m \in \mathfrak{D}} \phi\,(m) = 2\wp(\mathfrak{G})$. ☐

Theorem 3.4. *For any FG $\mathfrak{G}$ of order n, $\wp(\mathfrak{G}) = \wp_R(\mathfrak{G})$, if and only if, there is no strong edge between every vertex in $\mathfrak{G}$.*

Proof: Assume $g = (X_0, X_1, X_2)$ is a $\wp_R -$ function the equality $\wp(\mathfrak{G}) = \wp_R(\mathfrak{G})$ implies that we have;

$$\wp(\mathfrak{G}) \leq \sum_{m \in X_1 \cup X_2} \phi\,(m) = \sum_{m \in X_1} \phi\,(m) + \sum_{m \in X_2} \phi\,(m) \leq \sum_{m \in X_1} \phi\,(m) + 2\sum_{m \in X_2} \phi\,(m)$$

$$= \sum_{m \in X_1} \phi\,(m) + \sum_{m \in X_2} 2\,\phi(m) = \sum_{m \in X_1} g\,(m)\phi(m) + \sum_{m \in X_2} g\,(m)\phi(m) = \wp_R(\mathfrak{G})$$

Thus, $|X_0| = 0$ and $|X_2| = 0$, which implies that $|X_1| = n$. Since,
$\sum_{m \in X_2} \phi\,(m) = 2\sum_{m \in X_2} \phi\,(m)$, and $\sum_{m \in X_2} \phi\,(m) = 0$.
Therefore, $\wp_R(\mathfrak{G}) = \sum_{m \in X} \phi\,(m) = \wp(\mathfrak{G})$. ☐

Theorem 3.5. *For any complete FG (CFG), $\wp_R(\mathfrak{G}) = 2k$ such that $k = \min\phi(m){:}\, m \in X$.*

Proof: Suppose $\mathfrak{G} = (\phi, \psi)$ is a CFG of order n and $\phi(m) = \min\phi(m_0){:}\, m \in X$. Since in CFG, any pair of vertices are adjacent for a $\wp_R -$ function $g = (X_0, X_1, X_2)$, $|X_2| = 1$. So, if $X_2 = \{m_0\}$, such that $\phi(m_0) = \min\{\phi(m), m \in X\}$. Then, $\wp_R(\mathfrak{G}) = \sum_{m \in X} \phi\,(m) = 2k$. ☐

Theorem 3.6. *Suppose $g = (X_0, X_1, X_2)$ is a $\wp_R -$ function. Let $\mathfrak{G}$ is a FG, such that, for all $m \in X_1$, $\exists y, w \in \mathfrak{N}(m)$ and $\phi(m) < \phi(y) + \phi(w)$,*

(i) *Then, $\mathfrak{G}[X_1]$, the subgraph induced by X_1, $\forall m \in X_1$, $|\mathfrak{N}(m)| \leq 1$.*
(ii) *Each vertex of X_0 is adjacent to at most two vertices of X_1.*
(iii) *No edge of $\mathfrak{G}$ joins X_1 and X_2.*
(iv) *X_2 is a $\wp -$ set of $\mathfrak{G}[X_0 \cup X_2]$.*

Proof: (i) Suppose $m \in X_1$ is adjacent with two vertices $w, y \in X_1$, where $g(m) = g(y) = g(w) = 1$, therefore, $\wp_R = \phi(m) + \phi(y) + \phi(w)$. By reassigning $g(m) = 2$, $g(y) = g(w) = 0$ and keeping all other values of g to the same, we have $\wp_R = 2\phi(m)$, In this case, we find a new RDF with smaller weight, if $2\phi(m) \leq \phi(m) + \phi(y) + \phi(w)$, thus, $\phi(m) \leq \phi(y) + \phi(w)$.

(ii) Suppose $\forall m \in X_2$ is adjacent to all vertices $y, w, x \in X_0$ where, $g(m) = 0, g(y) = g(w) = g(x) = 0$. Therefore, $\wp_R = \phi(x) + \phi(y) + \phi(w)$. By reassigning $g(m) = 2$,

$g(y) = g(w) = g(x) = 0$ and keeping all other values of g to the same, we have $\wp_R = 2\phi(m)$, In this case, we find a new RDF with smaller weight. $2\phi(m) \leq \phi(x) + \phi(y) + \phi(w)$.

(iii) Suppose an edge joins $m \in X_2$ with $g(m) = 2$ and $y \in X_1$ with $g(y) = 1$. By reassigning $g(y) = 0$ and keeping all other values of g to the same, we find a new RDF with smaller weight, a contradiction.

(v)Assume $m \in X_0$ with $g(m) = 0$ is adjacent to at least one $y \in X_2$ with $g(y) = 2$. Therefore, X_2 is a DS of the subgraph induced by $X_0 \cup X_2$. ☐

Theorem 3.7. *Let $\mathfrak{G}^* = (X, E)$ and $\mathfrak{G}^{*'} = (X', E')$ are two FGs with $X = X'$ and $E \subseteq E'$. Then, $\wp_R(\mathfrak{G}^*) \leq \wp_R(\mathfrak{G}^{*'})$*

Proof: Observe that any RDF of $\mathfrak{G}^{*'}$ is an RDF of $\mathfrak{G}^*$. ☐

Theorem 3.8. *A FG $\mathfrak{G}$ is Roman if and only if has a $\wp_R -$ function $g = (X_0, X_1, X_2)$ with $|X_1| = n_1 = 0$.*

Proof: Suppose $\mathfrak{G}$ is an FG and $g = (X_0, X_1, X_2)$ is a $\wp_R -$ function of $\mathfrak{G}$. From Proposition 3.6(v) we know, the set X_2 dominates the set X_0 and the set $X_1 \cap X_2$ dominates the set X, thus

$$\wp(\mathfrak{G}) \leq \left|\sum_{m \in X_1} \phi(m)\right| + \left|\sum_{m \in X_2} \phi(m)\right| \leq \left|\sum_{m \in X_1} \phi(m)\right| + 2\left|\sum_{m \in X_2} \phi(m)\right| = \wp_R(\mathfrak{G})$$

Since $\mathfrak{G}$ is a Roman, we have,

$$2\wp(\mathfrak{G}) = 2\left|\sum_{m \in X_1} \phi(m)\right| + 2\left|\sum_{m \in X_2} \phi(m)\right| = \wp_R(\mathfrak{G}) = \left|\sum_{m \in X_1} \phi(m)\right| + 2\left|\sum_{m \in X_2} \phi(m)\right|$$

Therefore, $|X_1| = n_1 = 0$.

Conversely, suppose $g = (X_0, X_1, X_2)$ is a $\wp_R -$ function of $\mathfrak{G}$ with $|X_1| = n_1 = 0$. Therefore, $\wp_R(\mathfrak{G}) = 2\left|\sum_{m \in X_2} \phi(m)\right|$. Since $X_1 \cup X_2$ is adjacent with all vertex of $\mathfrak{G}$, it follows that X_2 is DS of $\mathfrak{G}$. Also, we know that X_2 is a $\wp -$ set of $\mathfrak{G}[X_0 \cup X_2]$. Therefore, $\left|\sum_{m \in X_2} \phi(m)\right| = \wp(\mathfrak{G})$ and $\wp_R(\mathfrak{G}) = 2\wp(\mathfrak{G})$. Hence, $\mathfrak{G}$ is a Roman graph. ☐

4. Conclusion

Graphs are simply models of relations. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to design an FG Model. In this paper, we have introduced the basic set-up of Roman domination in FGs. The existing research tends to focus on special properties of this idea and this paper serves to build a foundation for understanding various advanced problems. Also, we defined a dominating set, minimum dominating set and Roman domination function in an FG. Various results regarding the Roman domination of FGs are discussed.

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