

M.Sc. 1st Semester Examination, 2024**APPLIED MATHEMATICS***(Real Analysis)*

PAPER — MTM-101

*Full Marks : 50**Time : 2 hours*Answer **all** questions*The figures in the right hand margin indicate marks**Candidates are required to give their answers in their own words as far as practicable***GROUP — A**

1. Answer any *four* from the following questions : 2×4
- (a) Let $E \subseteq [a, b]$ be any set which has only a finite number of limit points. Can E be uncountable ?

- (b) Define σ -algebra with an example.
- (c) State Egoroff's theorem.
- (d) Show that every subset of a null set is a null set.
- (e) If α is continuous and is β of bounded variation on $[a, b]$, show that $\int_a^b \alpha d\beta$ exists.
- (f) For every $\epsilon > 0$ and $f \in L^1(\mu)$, show that

$$\mu\{x \in X : |f(x)| \geq \epsilon\} \leq \frac{1}{\epsilon} \int f d\mu.$$

GROUP – B

2. Answer any *four* from the following questions : 4 × 4
- (a) Establish the necessary and sufficient condition for a function $f : [a, b] \rightarrow \mathbb{R}$ to be a function of bounded variation on $[a, b]$.

(b) Let $f_n : X \rightarrow \mathbb{R}^*$ be measurable for $n = 1, 2, 3, \dots$. Then show that $\limsup_{n \rightarrow \infty} f_n$ and $\sup_{n \rightarrow \infty} f_n$ are measurable functions on X .

(c) If $f_n : X \rightarrow [0, \infty]$ is measurable for $n = 1, 2, 3, \dots$, and $f(x) = \sum_{n=1}^{\infty} f_n(x)$, $x \in X$, then show that $\int f \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$.

(d) Let μ be a measure on a σ -algebra of subsets of X . Show that the outer measure μ^* induced by μ is countably subadditive.

(e) Let $f_n : X \rightarrow [0, \infty]$ be measurable for $n = 1, 2, 3, \dots$,

$$f_1 \geq f_2 \geq f_3 \geq \dots \geq 0, \quad f_n(x) \rightarrow f(x)$$

as $n \rightarrow \infty$ for every $x \in X$, and $f_1 \in L^1(\mu)$.

Show that $\lim_{n \rightarrow \infty} \int f_n \, d\mu = \int f \, d\mu$.

- (f) Let α be a monotonically increasing function on $[a, b]$ and be continuous at x_0 where $a < x_0 < b$ and f be such that $f(x_0) = 1$ and $f(x) = 0$ for $x \neq x_0$. Show that f is R-S integrable with respect to α over $[a, b]$ and $\int_a^b f d\alpha = 0$.

GROUP – C

3. Answer any *two* questions : 8 × 2

- (a) (i) Check whether the function $f(x) = |5x - 1| + |x|$ on $[0, 3]$ is a function of bounded variation or not. If so, also find the variation function of f on $[0, 3]$. 4
- (ii) Show that the Cantor set is a null set. 4
- (b) (i) Suppose f is continuous on $[a, b]$ and α is monotonically increasing on $[a, b]$. Show that $f \in \mathcal{R}(\alpha)$ on $[a, b]$. 4

- (ii) Show that every finite sum of real numbers can be expressed as the R-S integral over some interval. 4
- (c) (i) Show that every bounded Riemann integrable function is Lebesgue integrable and the two integrals are equal in this case. 4
- (ii) Let $f: X \rightarrow [0, \infty]$ be measurable, $E \in \mathcal{M}$ and $\int_E f d\mu = 0$. Show that $f = 0$ a.e. on E . 4
- (d) (i) Let $f(x) = \frac{1}{7x^{\frac{5}{7}}}$ if $0 < x \leq 1$ and $f(0) = 0$. Show that f is Lebesgue integrable on $[0, 1]$ and find the value of the integral. 6

(ii) Evaluate the following : 2

$$\int_{-1}^3 7 \sin x \cdot x d(x - 3[x]).$$

[Internal Assessment – 10 Marks]
