M.Sc. 1st Semester Examination, 2024

APPLIED MATHEMATICS

(*Complex Analysis*)

PAPER – MTM-102

Full Marks : 50

Time : 2 hours

Answer **all** questions

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

GROUP – A

1. Answer any *four* questions : 2×4

(*a*) State the theorem for uniqueness of the Laurent's series representation.

(*Turn Over*)

(*b*) Find singular point/s of $\frac{1}{\sin\left(\frac{\pi}{2}\right)}$ and plots then in the complex plane, finally identify isolated and non-isolated singular points.

(*c*) Define branch and branch cut for a multi-valued function $f(z)$, and hence find these when $f(z) = \log(z)$.

(*d*) State the Jordan's lemma and also define the direct analytic continuation of a function.

(*e*) Using the argument principal, evaluate $\int_C \frac{f'(z)}{f(z)}\, dz$ when $f(z) = \frac{(z^2+1)^2}{(z^2+3z+2)^3}$ and $C : |z| = 3$, taken positive sense.

(*f*) Find whether $w = \frac{2z+1}{4z+2}$ is a bilinear transformation.

GROUP – B

2. Answer any *four* questions : 4 × 4

(*a*) Using the calculus of residue, evaluate

$$\int_0^\infty \frac{x^2}{1+x^6}\,dx.$$

(*b*) With the help of residue, find the inverse Laplace transformation $f(t)$ of

$$F(s)=\frac{1}{(s+a)^2+b^2}\ (a,b>0).$$

(*c*) Prove that the cross ratio remains invariant under the bilinear transformation.

(*d*) Show that if $f(z)$ is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)}{z-z_0}\,dz=\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

(*e*) Show that the Laurent's series expansion of $f(z)=\frac{1}{z^2-3z+2}$, valid in $0<|z-1|<1$, is of the form

$$\frac{1}{z^2-3z+2}=-\frac{1}{z-1}-\sum_{n=0}^{\infty}(z-1)^n.$$

(*f*) State and prove the Casorati-Weierstrass theorem.

GROUP – C

3. Answer any *two* questions : 8 × 2

(*a*) (*i*) Prove that all roots of $P(z)=z^7-5z^3+12=0$ lie between the circles $|z|=1$ and $|z|=2$.

(*ii*) State and prove the Cauchy's theorem to multiply-connected domain. 4 + 4

(*b*) (*i*) Using the method of residues and concept of integration along a branch cut, evaluate $\int_0^{\infty} \frac{x^{-a}}{x+1}\, dx$ where $0 < a < 1$.

(*ii*) Find the residue of $f(z) = \frac{1+z}{1-\cos z}$ at its singular point/s 6 + 2

(*c*) (*i*) State and prove the Liouville's theorem.

(*ii*) Find a conformal map of the unit disk $|z| < 1$ onto the right half-plane $\mathrm{Re}(w) > 0$. 4 + 4

(*d*) (*i*) Prove that a Mobius transformation $T = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$, is a holomorphic, bijective map from C onto C, and its inverse is also a Mobius transformation.

(*ii*) Write log(Log(i)) in the form $a + ib$, and also find its principal value. 6 + 2

[Internal Assessment — 10 Marks]