# M.Sc. 1st Semester Examination, 2024

## APPLIED MATHEMATICS

(*ODE and Special Functions*)

PAPER—MTM-103

*Full Marks* : 50

*Time* : 2 hours

Answer **all** questions

*The figures in the right hand margin indicate marks*

*Candidates are required to give their answers in their own words as far as practicable*

*Symbols have their usual meaning*

### GROUP – A

**1.** Answer any *four* questions : $2\times4$

(*a*) Define Green's function of the differential operator $L$ of the non-homogeneous differential equation :

$$Lu(x)=f(x).$$

(*b*) Examine that whether infinity is a regular singular point for Bessel's differential equation or not.

(*c*) Define fundamental set of solutions and fundamental matrix for system of ordinary differential equation.

(*d*) What do you mean by Wronskian in ODE and state its utility ?

(*e*) Let $P_n(z)$ be the Legendre polynomial of degree $n$ such that $P_n(1) = 1$, $n = 1, 2, 3, \ldots$

If $\int_{-1}^{1}\left(\sum_{j=1}^{n}\sqrt{j(2j-1)}P_j(z)\right)^2 dz = 20$,

then find the value of $n$.

(*f*) When a boundary problem is a Sturm-Liouville problem ?

**2.** Answer any *four* questions : $4 \times 4$

(*a*) Let $y_1(z)$ and $y_2(z)$ be two solutions of $(1-z^2)y''(z)-2zy'(z)+(\sec z)y=0$ with Wronskian $w(z)$. If $y_1(0)=1$, $w'(0)=0$ and $w\left(\frac{1}{2}\right)=\frac{1}{3}$, then find the value of $y'_2(z)$ at $z=0$.

(*b*) Deduce Rodrigue's formula for Legendre's polynomial.

(*c*) Establish generating function for Bessel's function.

(*d*) Prove that

$$\frac{d}{dz}[z^{-n}J_n(z)]=-z^{-n}J_{n+1}(z),$$

where $J_n(z)$ is the Bessel's function.

(*e*) Expand $z^4-3z^2+z$ in a series of form $\sum c_r P_r(z)$.

(*f*) If $m < n$, then show that

$$\int_{-1}^{1} z^n P_n(z)\,dz = \frac{2^{n+1}}{(2n+1)!}(n!)^2 .$$

**3.** Answer any *two* questions : 8 × 2

(*a*) (*i*) If α and β are the roots of the equation $J_n(z) = 0$ then show that 6

$$\int_0^1 J_n(\alpha z) J_n(\beta z)\,dz = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ \frac{1}{2}[J_n'(\beta)]^2 & \text{if } \alpha = \beta \end{cases}$$

(*ii*) If $z > 1$, then prove that 2

$$P_n(z) < P_{n+1}(z).$$

(*b*) (*i*) Prove that,

$$P_n(z) = P_{-n-1}(z).$$ 2

(*ii*) Find the general solution of the non-homogeneous system. 6

$$\frac{dX}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} X + \begin{pmatrix} -5t-6 \\ -4t+23 \\ 2 \end{pmatrix}$$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

(*c*) (*i*) Using Green's function method, solve the following differential equation

$$\frac{d^2y}{dx^2} - y = -2e^x \,;$$

$$y(0) = y'(0), y(l) + y'(l) = 0.$$ 6

(*ii*) Prove that

$$J_n(u+v) = \sum_{r=-\infty}^{\infty} J_r(u) J_{n-r}(v),$$

where $n$ is an integer. 2

(*d*) (*i*) Find the characteristics values and characteristic functions of the Sturm-Liouville problem 5

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right)+\frac{\lambda}{x}y=0;\ y'(1)=0,\ y'(e^{2\pi})=0.$$

(*ii*) Deduce the integral formula for hypergeometric function. 3

**[ Internal Assessment — 10 Marks ]**

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