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M.Sc. 3rd Semester Examination, 2024

MATHEMATICS

(*Partial Differential Equations and Generalized Functions*)

PAPER – MTM-301

Full Marks : 50

Time : 2 hours

Answer **all** questions

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

1. Answer any *four* questions from the following : 2×4

   (*a*) Define distribution with example.

   (*b*) Find the solution of $p^2z^2 + q^2 = 1$.

(*Turn Over*)

(*c*) Find the derivative of the Heaviside unit step function.

(*d*) Show that there is no maximum principle for the wave equation.

(*e*) Find the adjoint of the differential operator $L(u) = u_{xx} - 7u_{tt} + u_t$.

(*f*) Solve the following :

$(D^3 - 3D^2D' + 4D'^3)z = e^{2x+y}$ where

$D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$.

2. Answer any *four* questions from the following : 4×4

(*a*) Solve :

$$(x^2D^2 - xyDD' - 2y^2D'^2 + xD - 2yD')u = \log\left(\frac{y}{x}\right) - \frac{1}{2}.$$

Symbols have their usual meaning. 4

(*b*) Show that the type of a linear second order partial differential equation is invariant under a change of coordinates. 4

(*c*) Let $u \in C^2(D)$ be a function satisfying the mean value property in D. Show that u is harmonic function in D. 4

(*d*) Solve the following initial boundary value problem using parallelogram identity

$$u_{tt} - u_{xx} = 0,\ 0 < x < \infty,\ 0 < t < 2x$$
$$u(x, 0) = f(x),\ 0 \le x < \infty$$
$$u_t(x, 0) = g(x),\ 0 \le x < \infty$$
$$u(x, 2x) = h(x),\ x \ge 0,$$

where $f, g, h \in C^2[(0,\infty)]$. 4

(*e*) Let $u(x, t)$ be a solution of the equation $u_{tt} - u_{xx} = 0$ in the whole plane. Suppose that $u_x(x, t)$ is constant on the line $x = 1 + t$, Assume also that $u(x, 0) = 1$ and $u(1, 1) = 3$. Find such a solution $u(x, t)$. Is this solution unique? 4

(*f*) Show that the Green function for the Laplace equation is symmetric. 4

3. Answer any *two* questions from the following: 8 × 2

(*a*) (*i*) Solve:

$$\left(D^2 D' + D'^2 - 2\right) z = e^{2y} \cos 3x + e^x \sin 2y$$

where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$. 4

(*ii*) Use energy method to show that the following heat conduction problem has unique solution:

$$u_t - k u_{xx} = F(x, t),\ 0 < x < L,\ t > 0$$
$$u(0, t) = a(t),\ u(L, t) = b(t),\ t \ge 0$$
$$u(x, 0) = f(x),\ \ \le x < L.$$ 4

(*Continued*)

(*b*) (*i*) State and prove the weak minimum principle. 3

(*ii*) Establish the Poisson's formula for the solution of a Dirichlet problem for the Laplace equation in a disk of radius a. 5

(*c*) (*i*) Find the general solution of the problem

$$u_{ttx} - u_{xxx} = 0\ ,\ u_x(x, 0) = 0,\ u_{xt}(x, 0) = \sin x,$$

in the domain

$$\{(x, t) : | -\infty < x < \infty\ ,\ t > 0\ \}.$$ 3

(*ii*) Establish the d'Alembert's formula of the Cauchy problem for the non-homogeneous wave equation. 5

(*d*) Consider the following equation :

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0.$$

(*i*) Find the canonical form of the above equation. 5

(*ii*) Find the general solution $u(x, y)$ of the equation. 3

[Internal Assessment — 10 Marks]