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Determination of Hamiltonian Circuit and Euler Trail from a Given Graphic Degree Sequence

Krishnendu Basuli[1], Saptarshi Naskar[2] and Samar Sen Sarma[3]
Department of Computer Science and Engineering,
University of Calcutta, 92, A.P.C. Road,
Kolkata – 700 009, INDIA.
[1]krishnendu.basuli@gmail.com, [2]sapgrin@gmail.com, [3]sssarma2001@yahoo.com

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ABSTRACT

A sequence of nonnegative integers can represent degrees of a graph. Already there is an established condition under which the above conclusion is true. It is our objective in this paper to show that a similar degree sequence represents degrees of vertices of a graph having Hamiltonian Circuit or Euler Tour. The only fallacy of the conclusion is that there may be graphs those are not Hamiltonian or Euler graphs may have the same degree sequence.

Keywords: *Hamiltonian Circuit, Euler Trail, Graphic Sequence, Spanning Subgraph.*

1. Introduction

A sequence $\xi = d_1, d_2, d_3, \ldots, d_n$ of nonnegative integers is called a degree sequence of given graph G if the vertices of G can be labeled $V_1, V_2, V_3, \ldots, V_n$ so that degree $V_i = d_i$; for all $i=1,2,3,\ldots n$ [2]. For a given graph G, a degree sequence of G can be easily determined [1,3]. Now the question arise, given a sequence $\xi = d_1, d_2, d_3, \ldots, d_n$ of nonnegative integers, then under what conditions does there exist a graph G? A necessary and sufficient condition for a sequence to be graphical was found by Havel and later rediscovered by Hakimi [1,2,3]. As a sequel of this another question arises, if the sequence $\xi = d_1, d_2, d_3, \ldots, dn$, be a graphic sequence then is there any condition for which we can say that ξ represents a Hamiltonian graph sequence or Euler graph sequence? We have proposed an algorithm that accepts a graphic sequence and definitely determines whether at least Hamiltonian graph or an Euler graph can be drawn for the given graphic degree sequence or not.

2. Preliminaries

Definition: A sequence $\xi = d_1, d_2, d_3, \ldots, d_n$ of nonnegative integers is said to be *graphic sequence* if there exists a graph G whose vertices have degree d_i and G is called *realization* of ξ [1].

Definition: For any graph G, we define

$$\delta(G) = \min\{ \deg(v) \mid v \in V(G)\} \text{ and}$$

$$\Delta(G) = \max\{ \deg(v) \mid v \in V(G)\}.$$

If all the vertices of G have the same degree d then $\delta(G) = \Delta(G) = d$ and in this case the graph G is called the *Regular graph* of degree d [1].

Definition: A *Hamiltonian circuit* (or *Hamiltonian Cycle*) in a graph G is a circuit which contains every vertex of G [3]. A graph G is called *Hamiltonian Graph* if it contains a *Hamiltonian* cycle [3].

Definition: A *trail* in G is called an *Euler Trail* if it includes every edge of G [3].

Definition: A *Spanning Subgraph* G' of G is a subgraph containing all the vertices of G [4,5,6].

Theorem 1 (Dirac, 1952): If G is a graph with $n \geq 3$ vertices and $\delta \geq n/2$, then G is Hamiltonian [1,3].

Theorem 2: A degree sequence of graph G is given by $\xi = d_1, d_2, d_3, \ldots, d_n$. The graph is said to be Hamiltonian if and only if the degree sequence of the graph G can be represented as $\xi' = 2, 2, 2, \ldots, n^{th}$ term, where $n \geq 3$.

Proof: Hamiltonian circuit is a closed path, where each vertex papers once. Hence in a Hamiltonian Circuit degree of each vertex is exactly two.

Let us consider a given degree sequence ξ of the graph G. Now, deletion of a edge from a graph causes decrease of degrees of two vertices by unity.

Assuming, $\xi = d_1,d_2,d_3, \ldots,d_n$. and $d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n > 1$. If we can construct $\xi'=2,2,2,\ldots,n^{th}$ term from ξ, that means ξ' represents the degree sequence of the spanning subgraph of G. Hence there must be a subgraph of G which is a circuit containing all the vertices of G, which is nothing but the Hamiltonian circuit of the graph G. Hence, the theorem is proved.

Theorem 3: A connected graph G is Euler iff the degree of every vertex is even [3].

Theorem 4: A connected graph G has an Euler Trail iff it has at most two odd degree vertices, i.e. it has either no vertex of odd degree or exactly two vertices of odd degrees [3].

3. Proposed Algorithm: HAMILTON_CKT

Input: Sequence array.
Output: Contains Hamiltonian Circuit or not..
Step 1:

$\xi^{(0)} = d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n > 1$
IF ($d_n = 1$) Then
Stop.
End If

Step 2:

$k = d_1 - 1; d_1 = 2$
$d_2 = d_2 - 1, d_3 = d_3 - 1, \ldots , d_{k-1} = d_{k-1} - 1;$

Step 3:

Sort the sequence to make

$\xi^{(i)} = d_1^{(i)} \geq d_2^{(i)} \geq d_3^{(i)} \geq \ldots \geq d_n^{(i)}$

Step 4:

If ($d_1 = d_2 = d_3 = \ldots = d_n = 2$) Then
  Jump to Step 5
Else
  If (dn = 1) Then
    Jump to Step 6
  Else
    Jump to Step 2
  End If
End If

Step 5:

Print "The Graph is Hamiltonian Graph"
  Stop.

Step 6:

Print "The Graph is not a Hamiltonian Graph"

Step 7: Stop.

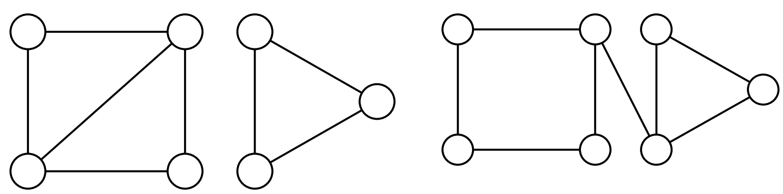
3.1. Explanation of the Algorithm HAMILTON_CKT with an Example

Let a degree sequence is ξ = 3, 3, 2, 2, 2, 2, 2. We have to check that the degree sequence represents a Hamiltonian graph or not.

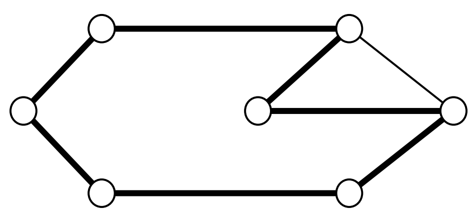
Since, $n = 7$, and $d_7 \neq 1$ we go for the next Step.

k = 3–2 = 1; d1 = 2, d2= 3–1=2

Now, ξ' = 2, 2, 2, 2, 2, 2, 2. Here, $d_1 = d_2 = d_3 = \ldots = d_n = 2$ satisfied. Hence, we stop here.



1.(a) Not a Hamiltonian graph   1.(b) Not a Hamiltonian graph



1.(c) A Hamiltonian graph

Figure 1:

Figure 1, shows that graphs having same degree sequence but some of them are not Hamiltonian graph. In Figure 1.(c) shows a Hamiltonian graph (Bold line indicates the circuit).

4. Proposed Algorithm: EULER_TOUR

Input: Sequence array.
Output: Represents Euler graph or not.
Step 1:

$\xi^{(0)} = d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n$

Step 2:

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If (∀ d_i is even numbers OR ξ contains exactly two odd numbers) Then
    Print "The sequence represents a Euler graph"
Else
    Print "The sequence represents a Euler graph"
End If
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Step 3:

Stop.

5. Conclusion

The algorithms we proposed actually identifies that the given graphic sequence represents any Hamiltonian graph or an Euler graph or not. Any two isomorphic graphs represent the exactly same sequence. However, the converse is not true [1]. So, the proposed theorems HAMILTON_CKT and EULER_TOUR actually says that if the given graphic sequence is a sequence of a Hamiltonian Graph or Euler Graph respectively then at least one Hamiltonian Graph and an Euler Graph can be possible for the sequence. Three non-isomorphic graphs, among them one is Hamiltonian Graph, but they represent the same degree sequence (shown in the Figure 1).

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