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Eccentricity Properties of Glue Graphs

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ABSTRACT

For any graph G, the Equi-eccentric point set graph G_{ee} is defined on the same set of vertices by joining two vertices in G_{ee} if and only if they correspond to two vertices of G with equal eccentricities. The Glue graph G_g is defined on the same set of vertices by joining two vertices in G_g if and only if they correspond to two adjacent vertices of G or two adjacent vertices of G_{ee}.

In this paper, radius and diameter of Glue graphs are studied. Eccentricity of each vertex of a glue graph is analyzed. Bounds for radius and diameter of a glue graph are given. Also, radius and diameter of glue graphs under certain conditions are found.

Keywords: *Equi-eccentric point set graph, Glue graph.*

1. Introduction

We consider only finite undirected graphs without loops and multiple edges. Let V(G) and E(G) denote the vertex set and edge set of G respectively. Eccentricity of a vertex $u \in V(G)$ is defined as $e(u) = \max \{d(u, v) : v \in V(G)\}$, where d(u, v) is the distance between u and v in G. The minimum and maximum eccentricities are the radius r and diameter d of G. When d(G) = r(G), G is called a self-centered graph with diameter d or r. A vertex u is said to be an eccentric point of v, when d(u, v) = e(v). In general, u is called as an eccentric point, if it is an eccentric point of some vertex, otherwise non-eccentric. Let E_k denote the set of vertices of G with eccentricity k and $|E_k| = c_k$, the cardinality of E_k. We have $c_r \geq 1$ and $c_i > 1$, i = r+1, r+2, ..., d.[2]. The definitions and details not furnished here may be found in Buckley and Harary [2].

For any graph G, the ***equi-eccentric point set graph*** G_{ee} is a graph with vertex set V(G) and two vertices are adjacent if and only if they correspond to two vertices of G with equal eccentricities. ***The Glue graph*** G_g of G is a graph with the same vertex set V(G) and two vertices are adjacent if and only if they correspond to two adjacent vertices of G_{ee} or two adjacent vertices of G.

The importance of perfect graphs is both theoretical and practical because of their application to perfect channels in communication theory, problems in operations research, optimizing municipal services etc. The Glue graph G_g is Hamiltonian and perfect. Also, G is a spanning subgraph of G_g and connectivity of

G_g increases and diameter of G_g decreases as that of G. Hence, these graphs will be useful in communication theory.

Now, let us state some important properties of the graph G.

(1) If G has a unique central vertex it may not be adjacent to every vertex of eccentricity r+1, but the central vertex must be adjacent to at least two vertices of eccentricity r+1.

(2) The induced sub graph formed by the vertices in E_{r+i}, i ≥ 1 is not complete.

(3) A vertex in E_{r+i} is not adjacent to every vertex of E_{r+i+1}.

(4) Any vertex with eccentricity k is adjacent to a vertex with eccentricity k or adjacent to a vertex with eccentricity k–1 or adjacent to a vertex with eccentricity k+1.

(5) $|E_r| \geq 1$ and $|E_k| \geq 2$ for all k > r.

(6) There exist at least two edges connecting vertices of E_k to vertices of E_{k+1} in G, k ≥ r.

(7) There exist at least two vertices in each E_{r+i}, which are adjacent to vertices of E_{r+i+1} and there exist at least two vertices, which are adjacent to vertices of E_{r+i-1}, …, at least two vertices in E_d, which are adjacent to vertices of E_{d-1}.

(8) A vertex in E_{r+1} may be adjacent to every vertex of E_r.

Now, let G be a tree with radius r and diameter d (d = 2r or 2r–1). G is n = d–r+1 eccentric. If d–r+1 is even, then in G_g all the vertices in E_r, E_{r+1}, …, $E_{(r+d)/2}$ have some of their eccentric points in E_d, and all the vertices in $E_{\lceil (r+d)/2 \rceil}$, $E_{\lceil (r+d)/2 \rceil +1}$,…, E_d have some of their eccentric points in E_r. If d–r+1 is odd, then in G_g all the vertices in E_r, E_{r+1}, …, $E_{[(r+d)/2]-1}$ have some of their eccentric points in E_d and all the vertices in $E_{[(r+d)/2]+1}$, …, E_d have some of their eccentric points in E_r and vertices in $E_{(r+d)/2}$ have their eccentric points in E_r or in E_d.

2.Main Results

First, we shall find out the exact values of radius and diameter of G_g when G is a tree.

Theorem 2.1 If T is a tree with diameter d and radius r, diameter of T_g is r and radius of T_g is (r+1)/2 or (r+2)/2.

Proof: **Case 1:** T is a uni-central tree.
Since T is uni-central d(T) = 2r.

Sub case 1.1: r is even.
T is n eccentric, where n = d–r+1 = r+1, since d = 2r. Let v_r be the central vertex. In T_g, eccentric point $\overline{v}_r$ of v_r lies in E_d, and $d(v_r, \overline{v}_r) = r$ in T_g. Hence, $e(v_r)$ in T_g is r. Let $v_{r+1} \in E_{r+1}$, In T_g, eccentric point $\overline{v}_{r+1}$ of v_{r+1} is in E_d and $d(v_{r+1}, \overline{v}_{r+1}) = r$ in T_g. [$\overline{v}_{r+1}$ is the eccentric point of v_{r+1} in T, then there exists a path $v_{r+1}\ v_r\ v_{r+1}'\ v_{r+2} \ldots \overline{v}_{r+1}$, where $v_{r+1}' \in E_{r+1}$, which is not adjacent to v_r in T. Hence, $v_{r+1}\ v_{r+1}'\ v_{r+2} \ldots \overline{v}_{r+1} = v_d$ is a shortest path in T_g]. Therefore, $e(v_{r+1}) = r$ in T_g. Similarly, we can prove that in T_g, $e(v_{r+2}) = r-1$, where $v_{r+2} \in E_{r+2}$, $e(v_{r+3}) = r-2$, where $v_{r+3} \in E_{r+3}$ ... , $e(v_{r+r/2}) = r-(r/2-1) = (r+2)/2$,

where $v_{r+r/2} \in E_{r+r/2}$, … , $e(v_d) = e(v_{2r}) = r$, where $v_d \in E_d$. Also in T_g, the maximum eccentricity is obtained for elements in E_r, E_{r+1}, E_d. The minimum eccentricity is obtained for elements in $E_{r+r/2}$. Hence, $d(T_g) = r$ and $r(T_g) = (r+2)/2$.

Sub case 1.2: r is odd.
Since r is odd $n = d-r+1$ is even. As in the previous case, we can prove in T_g, $e(v_r) = r$, $e(v_{r+1}) = r$, $e(v_{r+2}) = r-1$, …, $e(v_{r+(r-1)/2}) = r-[(r-1)/2-1] = (r+3)/2$,
$e(v_{r+(r+1)/2}) = r+(r+1)/2-r = (r+1)/2$, since in T_g, eccentric point of $v_k \in E_k$, $k > r+(r-1)/2$ is in E_r. $e((v_{r+(r+3)/2}) = r+(r+3)/2-r = (r+3)/2$, ..., $e(v_d) = e(v_{2r}) = 2r-r = r$.
Hence, $d(T_g) = r$ and $r(T_g) = (r+1)/2$.

Case 2: T is bi-central.
In this case, $d = 2r-1$, odd.

Sub case 2.1: r is even.
T has r eccentric sets. In T, no element of E_{r+1} is adjacent to all the elements of E_r, no element of E_{r+1} is adjacent to more than one element in E_r, since T is a tree. Let $E_r = \{v_r, v_r'\}$. Let $\bar{v}_r \in E_d$ be the eccentric point of v_r in T. Then the shortest path is $v_r\, v_r'\, v_{r+1}\, v_{r+2} \ldots v_d = \bar{v}_r$ in T. Hence in T_g, the shortest path is $v_r\, v_{r+1}\, v_{r+2} \ldots v_d = \bar{v}_r$. Therefore, $d(v_r, \bar{v}_r) = r$ in T_g. This gives $e(v_r) = r$ in T_g. Similarly, $e(v_r') = r$ in T_g. Now, take $v_{r+1} \in E_{r+1}$ in T, its eccentric point is in E_d. The shortest path in T is $v_{r+1}\, v_r\, v_r'\, v_{r+1}'\, v_{r+2} \ldots v_d = \bar{v}_{r+1}$. Hence, the shortest path in T_g is $v_{r+1}\, v_{r+1}'\, v_{r+2} \ldots v_d = \bar{v}_{r+1}$. Hence, $d(v_{r+1}, \bar{v}_{r+1}) = r-1$ in T_g. Therefore, $e(v_{r+1}) = r-1$ in T_g. Similarly, we can prove in T_g, $e(v_{r+2}) = r-2, \ldots, e(v_{r+(r/2-1)}) = r-(r/2-1) = (r+2)/2$. $e(v_{r+r/2}) = (r+2)/2$, since in this case eccentric point of $v_{r+r/2}$ in T_g is in E_r. (Since T is a tree, eccentric point of $v_{r+r/2}$ in T is in E_d, say v_d. Therefore, $v_{r+r/2}\, v_{r+r/2-1} \ldots v_{r+1}\, v_r\, v_r'\, v_{r+1}'\, v_{r+2}' \ldots v_{r+r/2}' \ldots v_d = \bar{v}_{r+r/2}$ is the shortest path in T and $d = 2r-1$ and hence in T_g, v_r' is the eccentric point of $v_{r+r/2}$ and $d(v_{r+r/2}, v_r') = r/2+1 = (r+2)/2)$. ..., $e(v_{d-1}) = r-1$, $e(v_d) = r$.
Hence, $d(T_g) = r$ and $r(T_g) = (r+2)/2$.

Sub case 2.2: r is odd.
Since r is odd, $n = d- r+1 = r$ is odd. As in the previous case, in T_g,
$e(v_r) = r$, $e(v_{r+1}) = r-1$, …, $e(v_{r+(r-1)/2}) = r-(r-1)/2 = (r+1)/2$, …,$e(v_d) = r$.
Hence, $d(T_g) = r$ and $r(T_g) = (r+1)/2$. Combining all the cases, we see that, $d(T_g) = r$ and $r(T_g) = (r+1)/2$ or $(r+2)/2$. This proves the theorem.

Now, let us find out the radius and diameter of G_g, when d is 2r or $2r-1$.

Theorem 2.2 Let G be a connected graph with radius r and diameter $d = 2r$, then $d(G_g)$ is r and $r(G_g)$ is $(r+1)/2$ or $(r+2)/2$.

Proof: $E_k = \{u \in V(G): e_G(u) = k\}$. G is n eccentric, where $n = d-r+1 = r+1$.
Case 1: r is odd.
Since $d = 2r$, every element of E_r lies on some diametral path joining two peripheral vertices at distance $d = 2r$ in G. Hence, $e(v_r) = r$ in G_g for all v_r in E_r. Now, consider, $v_{r+1} \in E_{r+1}$. In G_g, there exists a point $\bar{v}_{r+1}$, eccentric point of v_{r+1} lies in E_d and v_{r+1} is

not adjacent to every element of E_{r+2}. Hence, the shortest path from v_{r+1} to $\overline{v}_{r+1}$ in G is of the form $v_{r+1}\ v_r\ v_{r+1}'\ v_{r+2} \ldots v_d = \overline{v}_{r+1}$. Hence in G_g, the shortest path is $v_{r+1}\ v_{r+1}'\ v_{r+2} \ldots v_d = \overline{v}_{r+1}$ or $v_{r+1}\ v_{r+2}'\ v_{r+2} \ldots v_d = \overline{v}_{r+1}$. (Here v_{r+2}' is adjacent to v_{r+1} in G.) Therefore, $d(v_{r+1}, \overline{v}_{r+1}) = d-r = r$ in G_g. Hence, $e(v_{r+1}) = r$ in G_g. Similarly, we can prove that $e(v_{r+2}) = r-1$ in G_g. $e(v_{r+3}) = r-2, \ldots, e(v_{\lfloor (r+d)/2 \rfloor}) = r-[(r-1)/2-1] = (r+3)/2$, $e(v_{\lceil (r+d)/2 \rceil}) = (r+2)/2$, (since elements of $E_{\lceil (r+d)/2 \rceil}$ have their eccentric points in E_r)…, $e(v_d) = r$. Hence, $d(G_g)$ is r and $r(G_g)$ is (r+1)/2.

<u>Case 2:</u> <u>r is even.</u>
As in the previous case, in G_g
$e(v_r) = r$, $e(v_{r+1}) = d-r = r, \ldots, e(v_{(r+d)/2}) = d-\{r+[(r+d)/2]-1\} = (r+2)/2, \ldots, e(v_{d-1}) = r+1$, $e(v_d) = r$. Hence, $d(G_g)$ is r and $r(G_g)$ is (r+2)/2.
This proves the theorem.

Remark 2.1 If r is even and d = 2r, G_g is self-centered if and only if r = 2.

The next two theorems give the bounds for radius and diameter of G_g when G is a graph with radius r and diameter d.

Lemma 2.1 $d(G_g)$ is at most r.

Proof: Take v_r in E_r. For all x in V(G), $d(v_r, x) \le r$. Hence, in G_g also $d(v_r, x) \le r$ for all x in $V(G_g)$ and v_r in E_r. Now consider v_{r+1} in E_{r+1}. In G_g, $d(v_r, v_{r+1}) \le r$ for all v_r in E_r. Consider v_{r+2} in E_{r+2}. The shortest path from v_r in E_r to v_{r+2} in E_{r+2} contains at least one element from E_{r+1} and the length of the path is at most r. Also in G_g, any two elements of E_{r+1} are adjacent and hence $d(v_{r+1}, v_{r+2}) \le r$ for all $v_{r+2} \in E_{r+2}$ in G_g. Similarly, $d(v_{r+1}, v_{r+3}) \le r$ for all $v_{r+3} \in E_{r+3}$ in $G_g, \ldots, d(v_{r+1}, v_d) \le r$ for all $v_d \in E_d$ in G_g. Hence, $e(v_{r+1}) \le r$ in G_g. Similarly, we can prove that $e(v_{r+2}) \le r$, $e(v_{r+3}) \le r, \ldots, e(v_d) \le r$ for all $v_{r+2} \in E_{r+2}, v_{r+3} \in E_{r+3}, \ldots, v_d \in E_d$. Hence, diameter of G_g is at most r.

Theorem 2.3 $d-r \le d(G_g) \le r$.

Proof: Consider $v_r \in E_r$. In G_g, some eccentric points of v_r is in E_d and hence $d(v_r, v_d) \ge d-r$ for all $v_d \in E_d$. Hence, $e(v_r) \ge d-r$. Therefore, $d(G_g) \ge d-r$. By lemma 2.1 $d(G_g) \le r$. Thus, $d-r \le d(G_g) \le r$.

Remark 2.2 Upper bound in the above inequality for $d(G_g)$ is attained, when d < 2r also.

Theorem 2.4
(1) Radius of $G_g \le (3r-d+1)/2$ or $(3r-d+2)/2$.
(2) Radius of $G_g \ge (d-r)/2$ or $(d-r-1)/2$.

Proof: Let $v_r \in E_r$ and $v_d \in E_d$. In G, $d(v_r, v_d) \le r$ for all $v_r \in E_r$ and $v_d \in E_d$. $v_r \in E_r$ may be adjacent to some element of E_{r+1} or not. $v_{r+i} \in E_{r+i}$ may be adjacent to some element of E_{r+i+1}, E_{r+i} or E_{r+i-1}. But every path from v_r to v_d contains at least one element from $E_{r+1}, E_{r+2}, \ldots, E_d$. Let P be a shortest path from v_r to v_d.

Case 1: n = d–r+1 is odd.

In this case, d is even and r is even or d is odd and r is odd. P contains vertices from each E_k, k = r, r+1, …, d. Let P contains the point $v_{(r+d)/2} \in E_{(r+d)/2}$. Split the path P into two parts, path from v_r to $v_{(r+d)/2}$ and path from $v_{(r+d)/2}$ to v_d. Since P contains at least one vertex from each E_k, k = r, r+1, …, d, distance from v_r to $v_{(r+d)/2}$ is least when P contains exactly one vertex from each E_k, k = r+1, r+2,…, (r+d)/2. Therefore, $d(v_r, v_{(r+d)/2}) = d(v_r, v_{(r+(d-r)/2)}) \geq (d-r)/2$ and distance from v_r to $v_{(r+d)/2}$ is maximum when P contains exactly one vertex from each E_k, k = (r+d)/2+1, …, d. Hence, $d(v_r, v_{(r+d)/2}) \leq r-\{(d-r)-(d-r)/2\} = r-(d-r)/2 = (3r-d)/2$; that is $d(v_r, v_{(r+d)/2}) \leq (3r-d)/2$ when v_r and $v_{(r+d)/2}$ are points in P. Hence, in G_g for any $v_r \in E_r$ and $v_{(r+d)/2} \in E_{(r+d)/2}$, $d(v_r, v_{(r+d)/2}) \leq (3r-d)/2+1 = (3r-d+2)/2$ and $d(v_r, v_{(r+d)/2}) \geq (d-r)/2$. Similarly, we can prove that, in G_g, $d(v_d, v_{(r+d)/2}) \geq (d-r)/2$ and, $d(v_d, v_{(r+d)/2}) \leq (3r-d+2)/2$ for any $v_d \in E_d$ and $v_{(r+d)/2} \in E_{(r+d)/2}$. From the structure of G_g, it is clear that the central vertices of G_g belong to $E_{(r+d)/2}$ and their eccentric vertices are in E_r or in E_d. Hence, $(d-r)/2 \leq r(G_g) \leq (3r-d+2)/2$.

Case 2: n = d–r +1 is even.

In this case, d is even and r is odd or d is odd and r is even.

From the structure of G_g, it is clear that the central vertices of G_g belongs to $E_{\lfloor (r+d)/2 \rfloor}$ or $E_{\lceil (r+d)/2 \rceil}$ and $\lfloor (r+d)/2 \rfloor = (r+d-1)/2$, $\lceil (r+d)/2 \rceil = (r+d+2)/2$. As seen in case 1,
$d(v_r, v_{(r+d-1)/2}) \leq r-\{(d-r)-(d-r-1)/2\} = r-d+r+(d-r-1)/2 = (3r-d-1)/2$ and
$d(v_r, v_{(r+d+1)/2}) \leq r-\{(d-r)-(d-r+1)/2\} = (3r-d+1)/2$ for v_r, $v_{(r+d-1)/2}$, $v_{(r+d+1)/2}$, in the path P. Hence in G_g, $r(G_g) \leq (3r-d+1)/2$. Also for v_r, $v_{(r+d-1)/2}$, $v_{(r+d+1)/2}$ in P, $d(v_r, v_{(r+d-1)/2}) = d(v_r, v_{(r+(d-r-1)/2)}) \geq (d-r-1)/2$ and $d(v_r, v_{(r+d+1)/2}) \geq (d-r+1)/2$. Hence in G_g, $r(G_g) \geq (d-r-1)/2$. Thus, $(d-r-1)/2 \leq r(G_g) \leq (3r-d+1)/2$. This proves the theorem.

Next we give some upper bounds for radius and diameter of G_g, where G satisfies some conditions.

Theorem 2.5 If there exists vertices v_r in E_r, v_d in E_d such that there exists a path of length d–r from v_r to v_d in G, then radius of G_g is at most

$$\begin{cases} (d-r+4)/2 & \text{if } d-r \text{ is even.} \\ (d-r+5)/2 & \text{if } d-r \text{ is odd.} \end{cases}$$

Moreover, if G has a unique central vertex, then radius of G_g is at most

$$\begin{cases} (d-r+2)/2 & \text{if } d-r \text{ is even.} \\ (d-r+3)/2 & \text{if } d-r \text{ is odd.} \end{cases}$$

Proof: Let v_r' in E_r and v_d' in E_d such that there exists a path of length d–r from v_r' to v_d'. Hence in G_g, $e(v_r) \leq \max\{2, d-r+2\}$ for v_r in E_r; $e(v_{r+1}) \leq \max\{3, d-r+1\}$ for v_{r+1} in E_{r+1}; $e(v_{r+2}) \leq \max\{4, d-r\}$ for v_{r+2} in E_{r+2};……;
$e(v_{(r+d)/2}) \leq \max\{(d-r)/2+2, (d-r+2)-(d-r)/2\} = \max\{(d-r+4)/2, (d-r+4)/2\}$;
$e(v_{(r+d)/2+1}) \leq \max\{(d-r+2)/2+2, (d-r)-(d-r)/2+2\} = \max\{(d-r+6)/2, (d-r+4)/2\}$; …; $e(v_d) \leq \max\{d-r+1, 3\}$ for v_d in E_d. Thus, radius of $G_g \leq (d-r+4)/2$, if d–r is even. Similarly, we can prove that radius of $G_g \leq (d-r+5)/2$, if d–r is odd.

Suppose G is a graph with unique central vertex, then

Case 1: d–r is odd.

$e(v_{r+1}) \le \max\{2, d-r+1\}$ for v_{r+1} in E_{r+1}; $e(v_{r+2}) \le \max\{3, d-r\}$ for v_{r+2} in E_{r+2}; …;
$e(v_{\lfloor (r+d)/2 \rfloor}) \le \max\{(d-r+1)/2, (d-r+5)/2\}$ for $v_{\lfloor (r+d)/2 \rfloor} \in E_{\lfloor (r+d)/2 \rfloor}$;
$e(v_{\lceil (r+d)/2 \rceil}) \le \max\{(d-r+3)/2, (d-r+3)/2\}$ for $v_{\lceil (r+d)/2 \rceil} \in E_{\lceil (r+d)/2 \rceil}$; …;
$e(v_d) \le \max\{d-r+1, 2\}$ for v_d in E_d. Thus, radius of G_g is $\le (d-r+3)/2$.

Case 2: d–r is even.

$e(v_{(r+d)/2}) \le \max\{(d-r)/2, (d-r)-(d-r)/2+2\}$ for $v_{(r+d)/2} \in E_{(r+d)/2}$;
$e(v_{(r+d)/2+1}) \le \max\{(d-r+2)/2, (d-r+2)/2\}$ for $v_{(r+d)/2+1} \in E_{(r+d)/2+1}$;…;
$e(v_d) \le d-r+1$ for v_d in E_d. Thus, radius of G_g is $\le (d-r+2)/2$ if d–r is even.

This proves the theorem.

Now, assume that the graph G satisfies the following property A.

Property A: Every vertex with eccentricity k, where $k > r$ has adjacent vertices with eccentricity $k-1$ and $k+1$ and vertices in E_r have adjacent vertices in E_{r+1}.

In the following theorem we give bounds for radius and diameter of G_g, where G is a unicentral graph which satisfies the property A.

Theorem 2.6 Let G be a connected uni-central graph with radius r and diameter $d < 2r$, which satisfies property A, then diameter of G_g is $d-r+1$ and radius of G_g is $(d-r+3)/2$ or $(d-r+1)/2$ if d–r is odd, $(d-r+2)/2$ if d–r is even.

Proof: G is n-eccentric, where $n = d-r+1$.

Case 1: d–r is odd.

Sub case 1.1: There exists $v_d \in E_d$ such that $d_G(v_r, v_d) = d-r$ or there exists $v_{r+i} \in E_{r+i}$, where $i \ge (d-r+1)/2$ such that $d_G(v_r, v_{r+i}) = i$.

In G_g, some eccentric points of elements in $E_r, E_{r+1}, \ldots, E_{\lfloor (r+d)/2 \rfloor}$ are in E_d. Take $v_r \in E_r$. In G_g, $\bar{v}_r$ the eccentric point of v_r is in E_d. Since $d < 2r$, there is no shortest path like $v_r v_{r+1} v_{r+2} \ldots v_{d-1} v_d = \bar{v}_r$ in G. (Otherwise, $e(v_r) = d-r < r$.) Therefore, in G, the shortest path from v_r to $\bar{v}_r$ contains more than one element from some of the E_{r+i}'s. But in G_g, $< E_{r+I} >$ is complete and G satisfies property A. Hence in G_g, the shortest path from v_r to $\bar{v}_r$ is of the form $v_r v_{r+1} \ldots v_k' v_k \ldots v_d = \bar{v}_r$, where $v_k, v_k' \in E_k$. Hence, in G_g, $d(v_r, \bar{v}_r) = d-r+1$, $e(v_r) = d-r+1$ and $e(\bar{v}_r) = d-r+1$. Similarly, we can prove that $e(v_{r+1})$ in G_g is $d-r$; $e(v_{r+2})$ in G_g is $d-r-1$;…; $e(v_{\lfloor (r+d)/2 \rfloor}) = d-\{r+(d-r-1)/2-1\} = (d-r+3)/2$.

Now, consider $E_{\lceil (r+d)/2 \rceil}$. In G_g, elements of $E_{\lceil (r+d)/2 \rceil}$ have eccentric points in $E_r = \{v_r\}$. If the distance between $v_{\lceil (r+d)/2 \rceil} = v_{r+(d-r-1)/2}$ and v_r in G is $(d-r+1)/2$, then $e(v_{r+(d-r-1)/2})$ in G_g is $(d-r+1)/2$ (using the assumption), otherwise $(d-r+3)/2$. Therefore, $e(v_{\lceil (r+d)/2 \rceil}) = (d-r+1)/2$ or $(d-r+3)/2$, $e(v_{\lceil (r+d)/2 \rceil+1}) = (d-r+3)/2$, or $(d-r+5)/2$, ...,
$e(v_{d-1}) = d-r-1$ or $d-1$, $e(v_d) = d-r$ or $d-r+1$.
Hence, diameter of G_g is $d-r+1$ and radius of G_g is $(d-r+1)/2$.

Subcase 1.2: There exists no v_{r+i} in E_{r+i}, $i \geq (d-r+1)/2$ such that $d(v_r, v_{r+i}) = i$, in G.
In this case, $e(v_{\lceil (r+d)/2 \rceil}) = (d-r+3)/2$, $e(v_{\lceil (r+d)/2 \rceil +1}) = (d-r+5)/2$, ...,
$e(v_{d-1}) = d-r$, $e(v_d) = d-r+1$ in G_g.
Hence, diameter of G_g is d–r+1 and radius of G_g is (d–r+3)/2. Therefore, when d–r is odd, diameter of G_g is d–r+1 and radius of G_g is (d–r+1)/2 or (d–r+3)/2.

Case 2: d–r is even.

Sub case 2.1: There exists v_{r+i} in E_{r+i} such that $d(v_r, v_{r+i}) = i$ for all $i \geq (d-r+1)/2$ in G.
As in case 1, we can prove, in G_g, $e(v_r) = d-r+1$, $e(v_{r+1}) = d-r$, …,
$e(v_{(r+d)/2}) = (d-r+2)/2$, $e(v_{(r+d)/2+1}) = (d-r+2)/2$ or $(d-r+4)/2$, …,
$e(v_{d-1}) = d-r-1$ or $d-r$ and $e(v_d) = d-r+1$ or $d-r$.

Sub case 2.2: There exists no v_{r+i} in E_{r+i}, $i \geq (d-r+1)/2$ such that $d(v_r, v_{r+i}) = i$ in G.
In this case, in G_g, $e(v_r) = d-r+1$, $e(v_{r+1}) = d-r$,…, $e(v_{(r+d)/2}) = (d-r+2)/2$,
$e(v_{(r+d)/2+1}) = (d-r+4)/2$,…, $e(v_{d-1}) = d-r$, $e(v_d) = d-r+1$. Hence, in both the sub cases diameter of G_g is d–r+1 and radius of G_g is (d–r+2)/2. Therefore, when d–r is even, diameter of G_g is d–r+1 and radius of G_g is (d–r+2)/2.

This proves the theorem.

Remark 2.3 Suppose G is bi-eccentric uni-central with diameter d less than 2r, then d–r is odd ($r \geq 2$) and each element of E_{r+1} need not be adjacent to every element of E_r in G. If v_{r+1} in E_{r+1} is adjacent to all the elements of E_r, then in G_g, $e(v_{r+1})$ is one and for other elements of E_{r+1}, eccentricity is two in G_g. Hence, G_g is bi-eccentric with diameter two.

In the following theorem we give bounds for radius and diameter of G_g, where G is a graph with more than one central vertex and satisfies the property A.

Theorem 2.7 Let G be a graph with radius $r \geq 2$ and diameter $d < 2r$, having more than one central vertex with property A. Then diameter of G_g is d–r+1 and radius of G_g is (d–r+3)/2 if d–r is odd and (d–r+2)/2 if d–r is even.

Proof: It is given that $|E_r| = c_r \geq 2$. Therefore, v_r in E_r is not adjacent to at least one element of E_{r+1} and vice versa (since $d < 2r$ and G satisfies property A this is true).

Case 1: d–r is odd.
If n = 2, then G is bi-eccentric. In this case G_g is self-centered with diameter two. If n > 2, as in the previous theorem, in G_g, $e(v_r) = d-r+1$, $e(v_{r+1}) = d-r$,…, $e(v_{\lfloor (r+d)/2 \rfloor}) = (d-r+3)/2$, $e(v_{\lceil (r+d)/2 \rceil}) = (d-r+3)/2$,…, $e(v_{d-1}) = d-r$, $e(v_d) = d-r+1$. Therefore, diameter of G_g is d–r+1 and radius of G_g is (d–r+3)/2.

Case 2: d–r is even.

In G_g, $e(v_r) = d-r+1$, $e(v_{r+1}) = d-r, \ldots, e(v_{(r+d)/2}) = (d-r+2)/2, \ldots, e(v_{d-1}) = d-r$, $e(v_d) = d-r+1$. Therefore, diameter of G_g is $d-r+1$ and radius of G_g is $(d-r+2)/2$. This proves the theorem.

Eccentricity properties of $\overline{G}_g$

Next, we study the eccentricity properties of $\overline{G}_g$ for a connected graph G.

Let G be a connected (p, q) graph with radius r and diameter d. Let $n = d-r+1$. Then G is n eccentric.

<u>Case 1:</u> <u>$d-r+1 = 1$</u>.
In this case, G is self-centered, G_g is complete and hence $\overline{G}_g$ is totally disconnected.

<u>Case 2:</u> <u>$d-r+1 = 2$.</u>
In this case, G is bi-eccentric.

<u>Sub case 2.1:</u> <u>$r = 1$, $d = 2$.</u>
G_g is complete and hence G_g is totally disconnected.

<u>Sub case 2.2:</u> <u>$r > 1$.</u>
Here, different cases arise. If there exists $v_{r+1} \in E_{r+1}$ such that v_{r+1} is adjacent to all central vertices in G, then in $\overline{G}_g$, the point v_{r+1} is isolated. Similarly, if there exists $v_{r+1} \in E_{r+1}$ such that v_{r+1} is not adjacent to exactly one vertex of E_r in G and all other vertices in E_{r+1} are adjacent to v_r then $\overline{G}_g$ has K_2 as a component.

If G has only one central vertex, then it is adjacent to at least two vertices of E_{r+1} in G. Hence, those two vertices are always isolated in $\overline{G}_g$. Hence if G has a unique central vertex, $\overline{G}_g$ is disconnected. If $c_r = 2$, then let $E_r = \{v_r, v_r'\}$. There exists $v_{r+1}, v_{r+1}' \in E_{r+1}$ such that v_r is adjacent to v_{r+1} and v_r' is adjacent to v_{r+1}'. Suppose v_{r+1}' is not adjacent to v_r, there exists K_2 as a component, otherwise $\overline{G}_g$ has K_1 as a component. Therefore, c_r must be at least three for $\overline{G}_g$ to be connected. Also, each vertex of E_{r+1} is adjacent to at most c_r-2 vertices of E_r. When G satisfies these conditions $\overline{G}_g$ is connected and eccentricity of vertices of $\overline{G}_g$ lies between 2 and 4.

<u>Case 3:</u> <u>$n = d-r+1 = 3$.</u>
$V(G) = E_r \cup E_{r+1} \cup E_{r+2}$. In $\overline{G}_g$, $d(v_r, v_{r+2}) = 1$ for v_r in E_r and v_{r+2} in E_{r+2} and $d(v_r, v_{r+1}) \le 2$ since each v_{r+1} is not adjacent to at least one element in E_{r+2}. Also, $d(v_r, v_r') = 2$, since v_r, v_r' is adjacent to every element of E_{r+2}. Thus, in $\overline{G}_g$, $e(v_r) = 2$ for v_r in E_r. Also in $\overline{G}_g$, $d(v_{r+1}, v_{r+2}) \le 3$ if v_{r+1} and v_{r+2} are adjacent in G. (since $v_{r+1}\, v_{r+2}'\, v_r\, v_{r+2}$ is a path in $\overline{G}_g$) and $d(v_{r+1}, v_{r+1}') \le 4$ and $d(v_{r+1}, v_{r+1}') = 4$ if v_{r+1}, v_{r+1}' are adjacent to all the elements of E_r. Hence $e(v_{r+1}) \le 4$ in $\overline{G}_g$. Similarly, $d(v_{r+2}, v_{r+2}') \le 2$ and $d(v_{r+2}, v_{r+1}) \le 3$. Hence, $\overline{G}_g$ is connected and eccentricity of each vertex of $\overline{G}_g$ lies between 2 and 4. That is, $2 \le e(v) \le 4$ for $v \in V(\overline{G}_g)$.

<u>Case 4:</u> <u>$n = d-r+1 \ge 4$.</u>

In this case, eccentricity of each vertex in $\overline{G}_g$ is two. Hence, G_g is self-centered with diameter two, when $n \geq 4$.

Summarizing all these results, we obtain the following theorem.

Theorem 2.8 Let G be a connected (p, q) graph. Then (1) If G is self-centered or bi-eccentric with radius one then $\overline{G}_g$ is totally disconnected. (2) If G is bi-eccentric with radius greater than one and G contains at most two central vertices, then $\overline{G}_g$ is disconnected. (3) If G is bi-eccentric with radius greater than one and G has more than two central vertices then $\overline{G}_g$ is connected if and only if each vertex of E_{r+1} is adjacent to at most $c_r–2$ vertices of E_r. Also when $\overline{G}_g$ is connected, $2 \leq e(v) \leq 4$ for $v \in V(\overline{G}_g)$. (4) If G is tri-eccentric, then $\overline{G}_g$ is connected and $2 \leq e(v) \leq 4$ for $v \in V(\overline{G}_g)$. (5) If $n = d–r+1 \geq 4$, $\overline{G}_g$ is self-centered with diameter two.

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