Journal of Physical Sciences, Vol. 12, 2008, 41-49

Monotone Empirical Bayes Tests for the Parameter of a Positive Exponential Family Using PA Samples

Yi-min Shi and Shuan-hang Yan
Department of Applied Mathematics, Northwestern Polytechnical University,
Xi'an 710072, China
Email: ymshi@nwpu.edu.cn

Received October 27, 2008; accepted November 4, 2008

ABSTRACT

In this paper, we develop an empirical Bayes (EB) test by using the kernel-type density estimation in the case of identically distributed and positively associated (PA) samples. Given some suitable conditions, the asymptotically optimal property and the convergence rate of the EB test are obtained. Furthermore, it is proven that the convergence rate can arbitrarily approach to $O(n^{-1})$. Finally, an example about the main results of this paper is given. This example shows that the rationality of the conditions in theorem.

Keywords: *Empirical Bayes test, positively associated samples, asymptotic optimality, convergence rate.*

1. Introduction

We consider the Problem of testing the hypothesis $H_0 : \theta \le \theta_0 \leftrightarrow H_1 : \theta > \theta_0$ in a positive exponential family with the distribution density

$$f(x \mid \theta) = u(x)C(\theta)\exp(-\theta x)I_{(x>0)}, \theta > 0 \tag{1.1}$$

where θ_0 is a known positive constant, $u(x)$ is known positive and continuously and non- decreasing function for $x > 0$, $I(A)$ is the indicator of the set A. the sample space is $\Omega = \{x : x > 0\}$.

The loss function is defined to be

$$L(\theta, d_i) = (1-i)(\theta - \theta_0)I_{(\theta > \theta_0)} + i(\theta_0 - \theta)I_{(\theta \le \theta_0)}, i = 0,1 \tag{1.2}$$

where D={d_0,d_1} denotes action space with d_i accepting H_i.

Assume that the parameter θ has an unknown non-degenerate prior $G(\theta)$ with support on $\Theta = \{\theta > 0 : \int_\Omega f(x \mid \theta)dx = 1\}$. Hence the marginal probability density function (pdf) of random variable X is $f(x) = \int_\Theta f(x \mid \theta)dG(\theta) = u(x)\gamma_G(x)$, where

$\gamma_G(x) = \int_\Theta C(\theta)\exp(-\theta x)dG(\theta)$. Let

$\delta(x) = P(accept\ H_1 \mid X = x)$ be the randomized decision rule, then the Bayes risk of $\delta(x)$ is

$$R(\delta(x), G(\theta)) = \int_\Omega \int_\Theta \{L(\theta, d_1)\delta(x) + [1-\delta(x)]L(\theta, d_0)\} f(x \mid \theta)dG(\theta)dx$$

$$= \int_\Omega \delta(x)[\theta_0 - \beta_G(x)]f(x)dx + C_G = \int_\Omega \delta(x)u(x)m_G(x)dx + C_G$$

where $C_G = \int_\Theta (\theta - \theta_0)I_{(\theta > \theta_0)}dG(\theta)$, $\beta_G(x) = E(\theta \mid X = x) = \alpha_G(x)/\gamma_G(x)$

$$m_G(x) = \theta_0\gamma_G(x) - \alpha_G(x) \tag{1.3}$$

$$\alpha_G(x) = \int_\Theta \theta C(\theta)\exp(-x\theta)dG(\theta) \tag{1.4}$$

Since $\beta_G^{(1)}(x) = [E(\theta \mid X = x)]^2 - E(\theta^2 \mid X = x) = -Var(\theta \mid X = x) \le 0$ for all x, $\beta_G(x)$ is non-increasing and continuous function. We assume that $\lim_{x\to 0}\beta_G(x) > \theta_0 > \lim_{x\to\infty}\beta_G(x)$.

If the prior distribution $G(\theta)$ is non-degenerate, then under above assumption $\beta_G(x)$ is strictly decreasing in x. Thus there exists a point μ_G such that $\beta_G(\mu_G) = \theta_0, \beta_G(x) < \theta_0$ for $x > \mu_G$ and $\beta_G(x) > \theta_0$ for $x < \mu_G$. Therefore, the Bayes test δ_G can be written

$$\delta_G(x) = 1 \text{ if } x \le \mu_G \text{ and } \delta_G(x) = 0 \text{ if } x > \mu_G \tag{1.5}$$

where μ_G is called the critical point of the Bayes test δ_G. Thus, δ_G is a monotone Bayes test. However, Bayes test δ_G is unavailable to use since the prior $G(\theta)$ is unknown. As alternative we can use the empirical Bayes (EB)approach to obtain an EB test.

The EB approach has been studied extensively in the literature. For example literature [1],[2] discussed one-tail testing problem for the one-parameter continuous exponential family while [3]considered nonparametric EB solution to two-tail test in the exponential family. Also, literature [4~11] studied empirical Bayes test for the parameter for some distribution family. Differing from the past many works we consider the monotone empirical Bayes test for the parameter of a exponential family using PA samples.

The paper is organized as follows. In section 2, we proposed a monotone empirical Bayes test based on the NA samples. We investigated the asymptotic optimal and obtained a rate of convergence of order in Section3. An example with respect to the main result in this paper is given in Section 4.

2. Construction of EB Test

Firstly, we introduce the PA sequence which was first defined in [12].

Definition 1. Random variables $X_1,\cdots,X_n$ $(n\ge 2)$ are said to be positively associated (PA), if for every pair of disjoint nonempty subsets T_1 and T_2 of set $\{1,2,3,\cdots,n\}$, $Cov(f_1(X_i, i\in T_1), f_2(X_j, j\in T_2))\ge 0$, where f_1 and f_2 are increasing or decreasing for every variable such that this covariance exists. Random variable sequence $\{X_i\}$ $(i=1,2,3\cdots)$ are said to be PA, if for every natural number $n\ge 2$, $X_1,\cdots,X_n$, are positively associated.

In the empirical Bayes framework, let $X_1,\cdots,X_n,X_{n+1}$ $(X_{n+1}=X)$ are PA samples with the same marginal density function $f(x)$. Usually, we call $X_1,\cdots,X_n$ the past samples, X denotes the present sample. In this paper, $S_0$and S_1denote positive integers respectively, and $\{X_n : n\ge 2\}$ are identically distributed and weakly stationary PA sequence, $f(x)$ is a density function of X_1.The covariance of $\{X_n : n\ge 2\}$ satisfies condition:

(A) $\sum_{j=1}^{\infty}|Cov(X_1,X_j)|<\infty$

Based on $X_1,\cdots$, X_n and X, we define the estimation of $f(x)$ and $\alpha_G(x)$, $\phi_G(x)$, $\gamma_G(x)$, $m_G(x)$ respectively

$$f_n(x)=\frac{1}{nh_n}\sum_{i=1}^{n}K_0\left(\frac{X_i-x}{h_n}\right) \quad (2.1) \qquad \gamma_n(x)=\frac{1}{nh_n}\sum_{i=1}^{n}K_0\left(\frac{X_i-x}{h_n}\right)/u(X_i) \quad (2.2)$$

$$\alpha_n(x)=\frac{1}{nh_n^2}\sum_{i=1}^{n}K_1\left(\frac{X_i-x}{h_n}\right)/u(X_i) \quad (2.3) \qquad m_n(x)=\theta_0\gamma_n(x)-\alpha_n(x) \quad (2.4)$$

where $0<h_n\to 0$, $nh_n^6\to\infty$ $(n\to\infty)$, $K_i(x)$ is Borel-Measurable bounded function vanishing outside the interval $[0,1]$, $|K_i(x)|\le B$ for $x\in[0,1]$ $(i=1,2)$, and $K_i(x)$ satisfy following conditions:

(B) $\int_0^1 x^j k_i(x)dx=\begin{cases}1, & i=j,\\ 0, & i\ne j, j=0,1,\cdots,S_i-1,\\ B_i, & j=S_i.\end{cases}$

(C) $K(x)$ is differentiable on R_1 not including finite set E_0, $\sup_{x\in R_1-E_0}|k^{(1)}(x)|\le c<\infty$.

where B and B_i are constants .In the following, we assume that $G(\theta)\in\tau(r_1,r_2)=\{G: E(\theta)<\infty, \quad r_1\le\mu_G\le r_2\}$, where r_1,r_2 are known constants($0<r_1<r_2<\infty$). Then, it follows from the Bayes test $\delta_G(x)$ (1.5), and $r_1\le\mu_G\le r_2$,we propose the EB test as follows

$$\delta_n(x)=\begin{cases}1, & f(x\le r_1) \quad or(r_1\le x\le r_2 \; and \; m_n(x)\le 0,\\ 0, & if(x>r_1) \quad or(r_1\le x\le r_2 \; and \; m_n(x)>0.\end{cases} \tag{2.5}$$

Hence, the Bayes risk of $\delta_n(x)$ is $R(\delta_n(x),G(\theta))=\int_\Omega u(x)m_G(x)E[\delta_n(x)]dx+C_G$,where E denotes the expectation with respect to the joint distribution of $(X_1,\cdots,X_n)$ .By definition, an EB test $\delta_n(x)$ is said to be asymptotically optimal with respect to the prior distribution G, if $R(\delta_n(x),G(\theta))\to R(\delta_G(x),G(\theta))$, $(n\to\infty)$. If for some $q>0$, $R(\delta_n(x),G(\theta))-R(\delta_G(x),G(\theta))=O(n^{-q})$, then the convergence rate of the EB test $\delta_n(x)$ is denoted by (n^{-q}) .

3. The Properties of EB Test

In this paper, M , $M_0,M_1,\cdots$ always stand for some different positive constant that do not depend on n , they can take different values while appearing even within the same expression, and S S,S_0,S_1 denote different positive integer. R_i stand for i dimension real spaces. In order to obtain the proper- ties of EB test $\delta_n(x)$, we need the following lemmas.

Lemma 1. Let X and Y be PA variables with finite variance, then for any differentiable functions g_1 and g_2 we have

$$\left|\text{cov}(g_1(X),g_2(Y))\right|\le \sup_X\left|g_1^{(1)}(X)\right|\sup_Y\left|g_2^{(1)}(Y)\right|\text{cov}(X,Y) \tag{3.1}$$

When g_1 and g_2 are not differentiable on finite or countable set E_0^1 and E_0^2 respectively, We have

$$\left|\text{cov}(g_1(x),g_2(y))\right|\le \sup_{X\in R_1-E_0^1}\left|g_1^{(1)}(x)\right|\sup_{Y\in R_1-E_0^2}\left|g_2^{(1)}(Y)\right|Cov(X,Y) \tag{3.2}$$

Proof. The proof of (3.1) see lemma 4.2 of [13]. When g_1 and g_2 are not differentiable on finite or countable set E_0^1 and E_0^2 respectively, the integrate value of $\text{cov}(g_1(x),g_2(y))$ on product space $(R_1-E_0^1)\times(R_1-E_0^1)$ is equivalent on R_2 , so the proof of (3.2) is similarly to (3.1).

Lemma 2. Let $X_1,\cdots X_n$,$\cdots$, be identically distributed and weakly stationary PA sequence, $\gamma_n(x)$ and $\alpha_n(x)$ be defined by (2.2)and (2.3). If conditions (A)--(C) hold, $u(x)$ is non-decreasing function for all $x\in\Omega$, $\gamma_G(x)$ is continuous function and $\sup_{x>0}[\gamma_G(x)]=M_0<\infty$, then we have

(1) $Var[\gamma_n(x)] \le M(nh_n u(x))^{-1} + M(nh_n^4)^{-1}$ (3.3)

(2) $Var[\alpha_n(x)] \le M(nh_n^3 u(x))^{-1} + M(nh_n^6)^{-1}$ (3.4)

Proof. (1) $\mathrm{var}[\gamma_n(x)] = (nh_n)^{-2}\sum_{i=1}^{n} Var[K_0(\frac{X_i - x}{h_n})/u(X_i)]$

$$+2(nh_n)^{-2}\sum_{1\le i<j\le n} Cov(K_0(\frac{X_i-x}{h_n})/u(X_i), K_0(\frac{X_j-x}{h_n})/u(X_j) = T_1 + T_2 \quad (3.5)$$

Since $u(x)$ is non-decreasing function and $\gamma_G(x)$ is non-increasing function, $|K_0(x)| \le B$ and $\sup_{x>0}[\gamma_G(x)] = M_0 < \infty$, we have

$$T_1 = (nh_n)^{-2}\sum_{i=1}^{n} Var[K_0(\frac{X_i-x}{h_n})/u(X_i)] = (nh_n^2)^{-1} Var[K_0(\frac{X_1-x}{h_n})/u(X_1)]$$

$$\le (nh_n^2)^{-1} E[K_0(\frac{X_1-x}{h_n})/u(X_1)]^2 \le (nh_n u(x))^{-1}\int_0^1 K_0^2(v)\gamma_G(x+h_n v)dv \le M(nh_n u(x))^{-1} \quad (3.6)$$

Let $g_n(x,y) = K_0(\frac{x-y}{h_n})/u(y)$. From the condition (B), we know the partial derivative of $g_n(x,y)$ is existence on $R_1 - E_0$,then from lemma 1 and the condition (C), we have

$$T_2 = 2(nh_n)^{-2}\sum_{1\le i<j\le n} Cov(K_0(\frac{X_i-x}{h_n})/u(X_i), K_0(\frac{X_j-x}{h_n})/u(X_j)$$

$$\le 2(nh_n)^{-2}\sum_{1\le i<j\le n} |Cov(g_n(x,X_i), g_n(x,X_j))|$$

$$\le 2(nh_n)^{-2}\sum_{1\le i<j\le n}\{\sup_y(g_n(x,y)\}^2 Cov(X_i,X_j) \le M_1 n^{-2}h_n^{-4} n\sum_{i=1}^{\infty} Cov(X_i,X_j) \le M(h_n^4 n)^{-1} \quad (3.7)$$

Substituted (3.5)by (3.6)and (3.7), the proof of (1) is completed. Similarly, the proof of (2) can be completed.

Lemma 3. If the conditions of lemma 1 hold, and $\gamma_G(x)$ is the l times differentiable for all $x \in \Omega$. Then for all $l \ge 1$, we have

(1) $|E[\gamma_n(x)] - \gamma_G(x)| \le h_n^S Q(S,x)$, (2) $|E[\alpha_n(x)] - \alpha_G(x)| \le h_n^S Q(S+1,x)$,

(3) $|E[m_n(x)] - m_G(x)| \le h_n^S(\theta_0+1)Q(x)$.

where $Q(x) = \max(Q(S,x), Q(S+1,x))$, $Q(t,x) = B[(t+1)!]^{-1}\int_\Theta \theta^t C(\theta)\exp(-\theta x)dG(\theta)$.

Proof. By the Taylor's theorem and the condition (A), we have

$$E[\gamma_n(x)] = \int_0^1 K_0(v)\gamma_G(x+h_n v)dv = \int_\Theta C(\theta)\exp(-\theta x)\int_0^1 K_0(v)\exp(-\theta h_n v)dvdG(\theta$$

$$= \int_\Theta C(\theta)\exp(-\theta x)[1+\frac{(-h_n)^{S_0}\theta^{S_0}}{S_0!}\int_0^1 v^{S_0}K_0(v)\exp(-\theta h_n v)dv]dG(\theta)$$

$$= \gamma_G(x) + \frac{(-h_n)^{S_0}}{S_0!}\int_\Theta \theta^{S_0}C(\theta)\exp(-\theta x)[\int_0^1 v^{S_0}K_0(v)\exp(-\theta h_n v^*)dv]dG(\theta)$$

where $v^* \in [0,v]$, Since $|K_0(v)| \le B$ and $0 \le \exp(-\theta h_n v^*) \le 1$, we have

$$|E[\gamma_n(x)] - \gamma_G(x)| \le \frac{Bh_n^{S_0}}{(S_0+1)!}\int_\Theta \theta^{S_0}C(\theta)\exp(-\theta x)dG(\theta) = h_n^{S_0}Q(S_0,x)\,. \tag{3.8}$$

Similarly, since $|K_1(v)| \le B$ and $0 \le \exp(-\theta h_n v^{**}) \le 1$, where $v^{**} \in [0,v]$ .We have

$$|E[\alpha_n(x)] - \alpha_G(x)| = \left|\frac{(-h_n)^{S_1-1}}{S_1!}\int_\Theta \theta^{S_1}C(\theta)\exp(-\theta x)[\int_0^1 v^{S_1}K_1(v)\exp(-\theta h_n v^{**})dv]dG(\theta)\right|$$

$$\le \frac{Bh_n^{S_1-1}}{(S_1+1)!}\int_\Theta \theta^{S_1}C(\theta)\exp(-\theta x)dG(\theta) \le h_n^{S_1-1}Q(S_1,x) \tag{3.9}$$

Note that for each positive integer S , $Q(S_1,x)$ is decreasing in x . We choose $S_1 = S_0+1 = S+1$, from (3.8)and (3.9), we obtain

$$|E[\gamma_n(x)] - \gamma_G(x)| \le h_n^S Q(S,x)\,, \quad |E[\alpha_n(x)] - \alpha_G(x)| \le h_n^S Q(S+1,x)\,.$$

Define $Q(x) = \max(Q(S,x),Q(S+1,x))$ .Since both $Q(S,x)$ and $Q(S+1,x)$ are non-increasing function for $x>0$ □ $Q(x)$ is also non-increasing for $x>0$ .It follows that

$$|E[m_n(x)] - m_G(x)| \le \theta_0 h_n^S Q(S,x) + h_n^S Q(S+1,x) \le h_n^S(\theta_0+1)Q(x) \tag{3.10}$$

Since $u_G(x) \in (r_1,r_2)$, $Q(x)$ is non-increasing function for $x>0$ □ $Q(r_1) \ge Q(x)$ for all $x \ge r_1$. Thus
$\gamma_n(x)$, $\alpha_n(x)$ and $m_n(x)$ are asymptotically unbiased and consistent estimators of $\gamma_G(x)$, $\alpha_G(x)$ and $m_G(x)$ respectively. When n is larger enough, we have $\beta_G(r_1) - \theta_0 > 2h_n^S(\theta_0+1)Q(r_1)/\gamma_G(\mu_G)$, $\theta_0 - \beta_G(r_2) > 2h_n^S(\theta_0+1)Q(r_2)/\gamma_G(r_2)$. Since $\beta_G(x)$ is a continuous, strict decreasing function and $\beta_G(\mu_G) = \theta_0$, there exist two points μ_{G1} and μ_{G2}, $r_1 < \mu_{G1} < \mu_{G2} < r_2$ such that

$\beta_G(\mu_{G1}) - \theta_0 = 2h_n^S(\theta_0+1)Q(r_1)/\gamma_G(\mu_G)$, $\theta_0 - \beta_G(\mu_{G2}) > 2h_n^S(\theta_0+1)Q(r_2)/\gamma_G(r_2)$. (3.11)

Thus, we have $R(\delta_n(x), G(\theta)) - R(\delta_G(x), G(\theta)) = W1 + W2 + W3 + W4$, where

$$W1 = \int_{r_1}^{\mu_{G1}} P\{m_n(x) \le 0, m_G(x) > 0\} u(x) m_G(x) dx ,$$

$$W2 = \int_{\mu_{G1}}^{\mu_G} P\{m_n(x) > 0, m_G(x) \le 0\} u(x)\gamma_G(x)[\beta_G(x) - \theta_0] dx$$

$$W3 = \int_{\mu_G}^{\mu_{G2}} P\{m_n(x) \le 0, m_G(x) > 0\} u(x)\gamma_G(x)[\theta_0 - \beta_G(x)] dx ,$$

$$W4 = \int_{\mu_{G2}}^{r_2} P\{m_n(x) > 0, m_G(x) \le 0\} u(x)[-m_G(x)] dx$$

Theorem 1. Suppose that the EB test $\delta_n(x)$ is defined by (2.5), and $X_1, \cdots X_n, \cdots$, is identically distributed and weakly stationary PA sequence. If conditions (A)--(C) hold, $u(x)$ is a positive and continuous non-decreasing function for all $x \in \Omega$, and $E\theta < \infty$, $\gamma_G(x)$ is the l times differentiable for all $x \in \Omega$ $(l \ge 1)$ .Then, for each $G(\theta) \in \tau(r_1, r_2)$, when $\lim_{n\to\infty} h_n = 0, \lim_{n\to\infty} nh_n^6 = \infty$, we have

$$\lim_{n\to\infty} R(\delta_n(x), G(\theta)) = R(\delta_G(x), G(\theta))$$

If $h_n = n^{-1/(S+6)}$, then $R(\delta_n(x), G(\theta)) - R(\delta_G(x), G(\theta)) = O(n^{-S/(S+6)})$. Where $S \ge 1$, S is a positive integer.

Proof. Since $\gamma_G(x)$ is a decreasing function for all $x \in \Omega$, we have

$$\beta_G(\mu_{G1}) - \theta_0 \ge \beta_G(\mu_G) - \theta_0 = 0 \text{ for } x \in (\mu_{G1}, \mu_G), \quad (3.12)$$

$$0 \le \theta_0 - \beta_G(\mu_G) \le \theta_0 - \beta_G(x) \le \theta_0 - \beta_G(\mu_{G2}) \text{ for } x \in (\mu_G, \mu_{G2}) .$$

Thus, we have

$$W2 \le \int_{\mu_{G1}}^{\mu_G} 2u(x)\gamma_G(x) h_n^S(\theta_0+1)Q(r_1)/\gamma_G(\mu_G) dx \le 2h_n^S(\theta_0+1)Q(r_1)/\gamma_G(\mu_G) = O(h_n^S) ,$$

$$W3 \le \int_{\mu_G}^{\mu_{G2}} 2u(x)\gamma_G(x) h_n^S(\theta_0+1)Q(r_1)/\gamma_G(r_2) dx \le 2h_n^S(\theta_0+1)Q(r_1)/\gamma_G(r_2) = O(h_n^S) .$$

For all $x \in (r_1, \mu_{G1})$, we have $m_G(x) = \gamma_G(x)[\theta_0 - \beta_G(x)] < 0$. From lemma 3 and (3.11), we have

$$Em_n(x) < m_G(x) + (\theta_0+1)h_n^S Q(x)$$

$$\le \frac{1}{2}m_G(x) + \frac{1}{2}\gamma_G(\mu_G)[\theta_0 - \beta_G(\mu_{G1})] + (\theta_0+1)h_n^S Q(r_1) = \frac{1}{2}m_G(x) < 0 \quad (3.13)$$

For all $x \in (\mu_{G2}, r_2)$, we have $m_G(x) = \gamma_G(x)[\theta_0 - \beta_G(x)] > 0$ .Tthus by lemma 3 and (3.11), we have

$$Em_n(x) > m_G(x) - (\theta_0+1)h_n^S Q(x)$$

$$\ge \frac{1}{2}m_G(x) + \frac{1}{2}\gamma_G(r_2)[\theta_0 - \beta_G(\mu_{G2})] - (\theta_0+1)h_n^S Q(r_2) \ge \frac{1}{2}m_G(x) > 0 \qquad (3.14)$$

From lemma 2, we have

$$Var[m_n(x)] = Var[\theta_0\gamma_n(x) - \alpha_n(x)] \le 2\theta_0^2 Var[\gamma_n(x)] + 2Var[\alpha_n(x)]$$

$$\le M_0\gamma_G(x)/nh_n u(x) + M_1/nh_n^4 + M_2\gamma_G(x)/nh_n^3 u(x) + M_3/nh_n^6 = O(nh_n^6)^{-1} \qquad (3.15)$$

For each $x \in (\mu_{G2}, r_2)$, by (3.14),(3.15) and Tchebychev inequality, we have

$$W4 \le \int_{\mu_{G2}}^{r_2} P(|m_n(x) - Em_n(x)| > \frac{1}{2}m_G(x)) M(nh_n^6)^{-1}u(x)[-m_G(x)]dx$$

$$\le \int_{\mu_{G2}}^{r_2} \frac{4}{m_G^2(x)} M(nh_n^6)^{-1}u(x)[-m_G(x)]dx \le M(nh_n^6)^{-1}\int_{\mu_{G2}}^{r_2}[u(x)/m_G(x)]dx = O(nh_n^6)^{-1}$$

Similarly for $x \in (r_1, \mu_{G1})$ by (3.13), (3.15), we can get

$W1 \le M(nh_n^6)^{-1}\int_{r_1}^{\mu_{G1}} u(x)/m_G(x)dx = O(nh_n^6)^{-1}$.

When $0 < h_n \to 0 (n \to \infty), \lim_{n\to\infty} nh_n^6 = \infty$, we have

$$\lim_{n\to\infty} R(\delta_n, G) - R(\delta_G, G) = \lim_{n\to\infty}(W1 + W2 + W3 + W4) = 0$$

This shows that $\lim_{n\to\infty} R(\delta_n, G) = R(\delta_G, G)$. If $h_n = n^{-1/(S+6)}$, then

$R(\delta_n, G) - R(\delta_G, G) = O(n^{-S/6+S})$, where S is a positive integer $(S \ge 1)$.

Now, by using the conditions of theorem, we can show that if S is large enough, then the convergence rate of the empirical test also approaches to $O(n^{-1})$.

4. Example

Let $f(x|\theta) = \theta\exp(-\theta x)I(x>0)$, where $\theta > 0, u(x) = 1, C(\theta) = \theta$. The prior density function of θ is $g(\theta) = 4\theta\exp(-2\theta)I_{(\theta>0)}$ .Then by computation, we obtain that

$\gamma_G(x) = 8/(2+x)^3$, $\alpha_G(x) = 24/(2+x)^4$. Obviously, $u(x) = 1$ is a non-decreasing function for all $x > 0$, and $E\theta < \infty$, $\gamma_G(x)$ is the l times differentiable for all $x \in \Omega$ $(l \ge 1)$. Therefore, the conditions of theorem are satisfied. It is show that the results of this paper are obtained.

REFERENCES

1. Van Houwelingen, J.C., Monotone empirical Bayes tests for the continuous one-parameter exponential family, Ann. Statist., 4 (1976) 981-989.
2. Stijnen, T.H., On the asymptotic behaviour of empirical Bayes tests for the continuous one-parameter exponential family, Ann. Statist., 13 (1985) 403-412.
3. Zhang, S.P., Empirical Bayes test in multiple linear regression model, J. China University of Science and technology, 4 (1992), 498-506.
4. Liang,T.C., On empirical Bayes tests in a positive exponential family, J. Statist. Plann. Inference, 3 (2000), 169-181.
5. Liang, T.C., Monotone empirical Bayes tests for a discrete normal distribution, Staist. Prob. Lett., 4 (1999), 241-149.
6. Huang, S.Y, Empirical Bayes testing procedures in some non-exponential families using asymmetric Linex Loss function, J. Statist. Plann. Inference 46 (1995), 293-309.
7. Pan, J. M. On the Convergence rates in the central limit theorem for negatively associated sequences,Chinese Journal of appl. Prob. And stat., 2 (1977), 183-192
8. Wei, L.S., The empirical Bayes test problem for scale exponential family: in the case of NA samples, Acta Mathematicae Applicatae Sinica, 3 (2000), 403-412.
9. Chen L, Wei L S. The empirical Bayes tests problems for continuous one-parameter exponential family: In the Case of NA Samples. Mathematica Applicata, 17 (2004) 263-270.
10. Liang,T.C., on optimal convergence rate of empirical Bayes tests, Statistics & Probability Letter, 68 (2004) 189-198.
11. Wang lichun, Monotone empirical Bayes test for scale parameter under random censorship,4 (2007) 419-427.
12. Esary J.D.,Proschan F.,Walkup D.W. Association of Random Variables with Applications.Ann. Math. Statist, 38 (1967) 1466-1474.
13. Shao Q.M., Hao Yu. Weak Conbergence for weighted Empirical processes of dependent sequences, Statistics and Probability Letter, 50 (2000), 1-12.