M.Sc. 2nd Semester Examination, 2025

APPLIED MATHEMATICS

(*Abstract Algebra & Linear Algebra*)

PAPER – MTM-203(A & B)

Full Marks : 50

Time : 2 hours

Answer all questions

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

203A

(*Abstract Algebra*)

[*Marks* : 20]

Answer any two questions : 2×2

1. Show that $[\mathbf{R} : \mathbf{Q}]$ is not finite.

2. Is $\mathbb{R}$ an algebraic closure of $\mathbf{Q}$? Justify.

3. Determine the degree of

$$\left[\mathbf{Q}\left(\sqrt{3+2\sqrt{2}}\right):\mathbf{Q}\right].$$

4. Is it possible to square the unit circle ?

Answer any **two** questions : 4 × 2

5. Let $GF(p^n)$ be a Galois field of order p^n. Show that the mapping $f: GF(p^n) \to GF(p^n)$ defined by $f(a) = a^p$ for all $a \in GF(p^n)$ is an automorphism. Hence conclude that every finite field is a perfect field. 2 + 2

6. Let E be a field and G a finite group of automorphisms of E. Then show that E/E^G is a finite Galois extension.

7. State and prove the necessary and sufficient condition for a real number a to be constructible.

8. Show that $\left[Q\left(\sqrt{2},\sqrt{3}\right):Q\right]=4$.

Answer any **one** question : 8×1

9. (*i*) Let a regular polygon of n sides be constructible and p is an odd prime dividing n. Then show that p is a Fermat prime. 5

(*ii*) Show that $x^4+x^2+[1]$ is separable over $\mathbf{Z}_2$. 3

10. (*i*) Let K be a field. Show that there exists an algebraically closed field containing K. 4

(*ii*) If $K \subseteq F \subseteq L$ is a tower of fields then show that

$$[L : F]\,[F : K] = [L : K]$$

where $[L : F]$ denotes the degree of L over F. 4

[Internal Assessment — 05 Marks]

203B

(*Linear Algebra*)

[*Marks* : 20]

Answer any **two** of the following questions : 2×2

1. Let T be a linear operator on $P_2(\mathbf{R})$ defined by $T(f(x)) = f'(x)$. Check whether T is diagonalizable or not ?

2. Define minimal polynomial for a linear operator.

3. Find the matrix of the bilinear form

$$H(x, y) = x_1y_1 + 2x_1y_2 + 5x_1y_3 - 2x_2y_1 + x_2y_3 - 6x_3y_2 + 6x_3y_3.$$

4. Let $\lambda \in \mathbf{F}$ and $T : \mathbf{F}^2 \to \mathbf{F}^2$ be a linear operator defined by $T(x, y) = (2x + \lambda y, 3x + 7y)$ for all $(x, y) \in \mathbf{F}^2$. Find the value(s) of λ such that T is a self-adjoint linear operator.

Answer any **two** of the following questions : 4×2

5. Let $P_2(\mathbf{R})$ denotes the collection of all polynomials of degree ≤ 2 and $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be a linear operator defined by $T(f(x)) = 2f(x) - f'(x)$ for all $f((x) \in P_2(\mathbf{R})$. Find the Jordan canonical form of T.

6. Show that every bilinear form on a vector space V over the field F can be uniqualy, express as the sum of a symmetric bilinear form and a skew-symmetric bilinear form.

7. Prove that if V be an inner product space and let T be a normal operator on V, then the following statements are true.

   (*i*) $\|T(x)\| = \|T^*(x)\|$ for all $x \in V$.

   (*ii*) $T - cI$ is normal for every $c \in F$.

   (*iii*) If x is an eigenvector of T, then x is also an eigenvector of T^*. In fact, if $T(x) = \lambda x$, then $T^*(x) = \bar{\lambda} x$.

   (*iv*) If λ_1 and λ_2 are distinct eigenvalues of T with corresponding eigen-vectors x_1 and x_2, then x_1 and x_2 are orthogonal.

8. Let T be a linear operator on $\mathbb{R}^3$, which is represented in a standard ordered basis by the matrix

$$A = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}$$

Is A similar over the field $\mathbb{R}$ to a diagonal matrix ? Is A similar over the field $\mathbf{C}$ to a diagonal matrix ?

Answer any **one** of the following question : 8 × 1

9. (*i*) Using diagonalization to solve the system of differential equations :

$$x_1' = 3x_1 + x_2 + x_3$$
$$x_2' = 2x_1 + 4x_2 + 2x_3$$
$$x_3' = -x_1 - x_2 + x_3$$

(*ii*) Let T be a linear operator on an inner product space V, and suppose the $\|T(x)\| = \| x \|$ for all x. Prove that T is one-one. 5 + 3

10. (*i*) Suppose V is the vector space of all polynomials over the field $\mathbf{R}$ of degree ≤ 2. Let ϕ_1, ϕ_2, ϕ_3 be the linear functionals on V given by

$\phi_1(f(t)) = \int_0^1 f(t)dt$, $\phi_2(f(t)) = f'(t)$, for $t = 1$. $\phi_3(f(t)) = f(0)$.

Where $f(t) = a + bt + ct^2 \in V$ and $f'(t) = \frac{d}{dt} f(t)$. Find the basis $\{f_1(t), f_2(t), f_3(t)\}$ of V which is dual to $\{\phi_1, \phi_2, \phi_3\}$.

(*ii*) Compute the minimal polynomial of the following :

$$V = P_2(\mathbf{R}), \text{ and } T(f(x)) = -xf''(x) + f'(x) + 2f(x).$$

5 + 3

[Internal Assessment — 05 Marks]