PG/IVS/MTM/401/25

# M. Sc. 4th Semester Examination, 2025

## APPLIED MATHEMATICS

(*Functional Analysis*)

PAPER – MTM-401

*Full Marks* : 50

*Time* : 2 hours

Answer **all** questions

*The figures in the right hand margin indicate marks*

*Candidates are required to give their answers in their own words as far as practicable*

1. Answer any *four* questions : $2 \times 4$

(*a*) Let $X$ and $Y$ be normed spaces. If $X$ is finite dimensional, then show that every linear map from $X$ to $Y$ is continuous.

( *Turn Over* )

(*b*) State and prove the parallelogram law.

(*c*) Let $X$ be a normed space. Show that $x_n \to x$ weakly in X does not imply $x_n \to x$ in X in general.

(*d*) Give an example of a normal operator which is not self-adjoint.

(*e*) Let $T \in BL(H)$ be a normal operator. Then show that $Ker(T) = Ker(T^*)$.

(*f*) Let $X$ be a normed space and $f(x) = f(y)$ for every $f \in X^*$, show that $x = y$.

2. Answer any *four* questions : 4 × 4

(*a*) Let $X = C[0,1]$ with the supremum norm.

Consider the sequence $x_n(t) = \frac{t^n}{n^7}, t \in [0,1]$.

Check whether the series $\sum_{n=1}^{\infty} x_n$ is summable in $X$. 4

(*b*) Let $F \in BL(X, Y)$ and $Z(F) = \{x \in X : F(x) = 0\}$. Define $\tilde{F} : X / Z(F) \to Y$ by $\tilde{F}(x + Z(F)) = F(x), x \in X$. Show that $\|\tilde{F}\| = \|F\|$. 4

(*c*) Show that a Banach space is a Hilbert space if an only if the Parallelogram law holds. 4

(*d*) Let $P \in BL(H)$ be a non-zero projection on a Hilbert space $H$. Show that if $\|P\| = 1$, then $P$ is an orthogonal projection. 4

(*e*) Let $X$ be an inner product space and $A \subseteq X$. Then show that

(*i*) $A \subseteq A^{\perp\perp}$, (*ii*) $A^{\perp} = A^{\perp\perp\perp}$. 4

(*f*) Let $X$ and $Y$ be Banach spaces and $A \in BL(X, Y)$. Show that there is a constant

$c > 0$ such that $\|Ax\| \geq c\|x\|$ for all $x \in X$ if and only if $Ker(A) = \{0\}$ and $Ran(A)$ is closed in $X$. 4

3. Answer any *two* questions : $8 \times 2$

(*a*) (*i*) Let $H$ be a Hilbert space, $T \in BL(H)$ and $< Ax, x >= 0$ for all $x \in H$. Then show that $x = 0$. Also, give an example to show that this result is not true for real Hilbert space. 5

(*ii*) Let $X = C^1[0,1]$ be equipped with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $Y = C[0,1]$ be equipped with the supremum norm $\|x\|_\infty$. Check whether the linear operator $F : X \to Y$ defined by $F(x) = x$ is continuous. 3

(*b*) (*i*) Let the space $l^2(\mathbb{Z})$ be defined as the space of all two-sided square summable sequences and the bilateral shift is the operator W on $l^2(\mathbb{Z})$ defined by $W(\ldots, a_{-2}, a_{-1}, \hat{a}_0, a_1, a_2, \ldots) = (\ldots, a_{-3}, a_{-2}, \hat{a}_{-1}, a_0, a_1, \ldots)$. Prove that

(*I*) $W$ is unitary and

(*II*) the adjoint $W^*$ of $W$ is given by $W^*(\ldots, a_{-2}, a_{-1}, \hat{a}_0, a_1, a_2, \ldots) = (\ldots, a_{-1}, a_0, \hat{a}_1, a_2, a_3, \ldots)$. 5

(*ii*) State and prove Riesz-Fischer Theorem. 3

(*c*) (*i*) Let $H$ be a Hilbert space and $A \in BL(H)$. If $A$ is self-adjoint, then prove that $\|A\| = \sup\{|< Ax, x >| : x \in X, \|x\| = 1\}$. 5

(*ii*) Show that $Ran(T) = Ran(T^*)$ if $T \in BL(H)$ is normal and $H$ is a Hilbert space. 3

(*d*) (*i*) Let $X$ be an inner product space, $E \subseteq X$ be closed under scalar multiplication and $x \in X$. Show that $x \perp E$ if and only if $dist(x, E) = \|x\|$. 5

(*ii*) Let $S \in BL(H)$, where $H$ is a Hilbert space. Prove that for all $x, y \in H$,

$$< Sx, y > = \frac{1}{4}\sum_{n=0}^{3} i^n < S(x + i^n y), (x + i^n y) > .$$ 3

**[ Internal Assessment – 10 Marks]**

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