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New Numerical Algorithms for Unconstrained Optimization Problems

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ABSTRACT

Two new algorithms namely, Circle approach algorithm and Circle-tangent approach algorithm are proposed for unconstrained optimization problems. Then, comparative study among the new algorithms and Newton's algorithm are established by means of examples.

1. Introduction

Optimization problems with or without constraints arise in various fields such as science, engineering, economics, management sciences, etc., where numerical information is processed. In recent times, many problems in business situations and engineering designs have been modeled as an optimization problem for taking optimal decisions. In fact, numerical optimization techniques have made deep in to almost all branches of engineering and mathematics.

An unconstrained minimization problem is the one where a value of the vector x is sought that minimizes the objective function $f(x)$. This problem can be considered as particular case of the general constrained non-linear programming problem. The study of unconstrained minimization techniques provide the basic understanding necessary for the study of constrained minimization methods and this method can be used to solve certain complex engineering analysis problem. For example, the displacement response (linear or non-linear) of any structure under any specified load condition can be found by minimizing its potential energy.

Several methods [5,7,8,10,11,15,16]are available for solving unconstrained minimization problems. These methods can be classified in to two categories as non gradient and gradient methods. The non gradient methods require only the objective function values but not the derivatives of the function in finding minimum. The gradient methods require, in addition to the function values, the first and in some cases the second derivatives of the objective function. Since more information about the function being minimized is used through the use of derivatives, gradient methods are generally more efficient than non gradient methods. All the unconstrained minimization methods are iterative in nature and hence they start from an initial trial solution and proceed towards the minimum point in a sequential manner.

To solve unconstrained nonlinear minimization problems arising in the diversified field of engineering and technology, we have several methods to get solutions. For instance, multi- step nonlinear conjugate gradient methods [3], a scaled nonlinear conjugate gradient algorithm[1], a method called, ABS-MPVT algorithm [12] are used for solving unconstrained optimization problems. Newton's method [13] is used for various classes of optimization problems, such as unconstrained minimization problems, equality constrained minimization problems. A proximal bundle method with inexact data [17] is used for minimizing unconstrained non smooth convex function. An incremental method [4] is used for solving convex finite min-max problems. Implicit and adaptive inverse preconditioned gradient method [2] is used for solving nonlinear minimization problems. A new algorithm [6] is used for solving unconstrained optimization problem with the form of sum of squares minimization. A derivative based algorithm [9] is used for a particular class of mixed variable optimization problems. A globally derivative – free decent method [14] is used for nonlinear complementarity problems. Vinay Kanwar *et al.* [18] introduced new algorithms called, external touch technique and orthogonal intersection technique for solving the non linear equations. Further, they did the comparative study of the new algorithms and Newton's algorithm.

In this paper, we propose two new algorithms namely, Circle approach algorithm and Circle-tangent approach algorithm for minimizing non linear functions. Then, we present the comparative study among the new algorithms and Newton's algorithm by means of examples.

2. New algorithms

In this section , we introduce two new numerical algorithms namely, circle approach algorithm and circle-tangent approach algorithm for minimizing nonlinear real valued and twice differentiable real functions using the concept of external touch numerical algorithm [18].

Consider the nonlinear optimization problem :

$$\text{Minimize } \{f(x),\ x \in R,\ f : R \to R\}$$

where f is a non-linear twice differentiable function.

2.1 Circle approach algorithm

Consider the function $G(x) = x - (g(x)/g'(x))$ where $g(x) = f'(x)$.
Here $f(x)$ is the function to be minimized. $G'(x)$ is defined around the critical point x^* of $f(x)$ if $g'(x^*) = f''(x^*) \neq 0$ and is given by

$$G'(x) = g(x)g''(x)/g'(x).$$

If we assume that $g''(x^*) \neq 0$, we have $G'(x^*) = 0$ iff $g(x^*) = 0$.

Consider the equation

$$g(x) = 0 \qquad \text{--- (1)}$$

whose one or more roots are to be found. $y = g(x)$ represents the graph of the function $g(x)$ and assume that an initial estimate x_0 is known for the desired root of the equation $g(x) = 0$.

A circle C_1 of radius $g(x_0 + h)$ is drawn with centre at any point $(x_0 + h, g(x_0 + h))$ on the curve of the function $y = g(x)$ where h is small positive or negative quantity. Another circle C_2 with radius $g(x_0 - h)$ and centre at $(x_0 - h, g(x_0 - h))$ is drawn on the curve of the function g(x) such that it touches the circle C_1 externally.

Let $x_1 = x_0 + h$, $|h| < 1$.

Since the circles C_2 and C_1 touches externally, we have

$$h^2 = g(x_0 + h)g(x_0 - h).$$

Expanding $g(x_0 + h)$ and $g(x_0 - h)$ by Taylor's series (omitting fourth and higher powers of h) and simplifying, we can conclude that

$$h = \pm \frac{g(x_o)}{\sqrt{1 + g'^2(x_o) - g(x_o)g''(x_o)}} \qquad \text{---- (2)}$$

where h can be taken positive or negative according as x_0 lies in the left or right of true root or slope of the curve at $(x_0, g(x_0))$ is positive or negative. If x_0 lies in the left of true root, then h is taken as positive otherwise negative. Therefore, we get the first approximation to the root as $x_1 = x_0 \pm h$.

That is, $x_1 = x_0 \pm \dfrac{g(x_o)}{\sqrt{1 + g'^2(x_o) - g(x_o)g''(x_o)}}$.

Since $g(x) = f'(x)$, it follows that $x_1 = x_0 \pm \dfrac{f'(x_o)}{\sqrt{1 + f''^2(x_o) - f'(x_o)f'''(x_o)}}$

The general iteration formula for successive approximate minimizing point of the non-linear function f is

$$x_{n+1} = x_n \pm \frac{f'(x_n)}{\sqrt{1 + f''^2(x_n) - f'(x_n)f'''(x_n)}} \qquad \text{----(3)}$$

Note: The sufficient condition for convergence in the interval containing the root is given $f'(x_n)f'''(x_n) < 1 + f''^2(x_n)$

Now, we give the Circle approach algorithm for finding the minimizing point of the non-linear function of the real valued real functions

Algorithm:

1. Given a non-linear function $f(x)$
2. Find a and b such that $f'(a)$ and $f'(b)$ are of opposite signs such that $a < b$.
3. Input $x_0, \in, f'(x), f''(x), f'''(x)$
4. $n = 0$
5. If we choose $x_0 = a$, then go to step 6
   otherwise if we choose $x_0 = b$, then go to step12
6. Repeat
7. $x_{n+1} = x_n + \dfrac{f'(x_n)}{\sqrt{1 + f''^2(x_n) - f'(x_n)f'''(x_n)}}$
8. $n \leftarrow n+1$
9. Until $|x_n - x_{n-1}| < \in$
10. Optimal solution $x^* \leftarrow x_n$
11. End
12. Repeat
13. $x_{n+1} = x_n - \dfrac{f'(x_n)}{\sqrt{1 + f''^2(x_n) - f'(x_n)f'''(x_n)}}$
14. $n \leftarrow n+1$
15. Until $|x_n - x_{n-1}| < \in$
16. Optimal solution $x^* \leftarrow x_n$
17. End

Convergence analysis for Circle approach algorithm

This algorithm is similar to the external touch technique of circles [18]. Since the order of convergence of the external touch technique of circles is 2, we have the order of convergence this method is 2.

By Newton's algorithm and Circle approach algorithm, a minimizing point for the given function are obtained and then, the comparative study of the above said algorithms have been established by means of examples.

Example 1: Consider the function $f(x) = x^4 - x - 10 \;\; x \in R$. Then, minimizing point of function is equal to 0.62996 which is obtained in 6 iterations by Circle approach algorithm and in 8 iterations by Newton's algorithm (refer Table -1 and Figgure-1).

Table-1

Iterations	Newton's algorithm	Circle approach algorithm
1	3	3
2	2.00926	1.299873
3	1.36015	0.701412
4	0.95181	0.631332
5	0.72653	0.62999
6	0.64223	0.62996
7	0.63019	
8	0.62996	

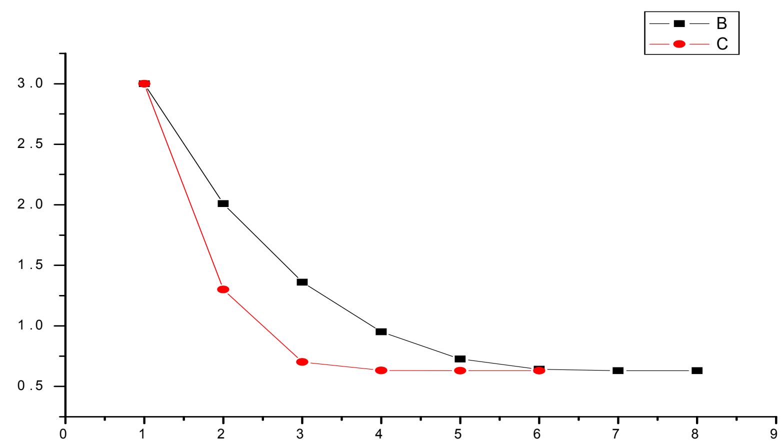

Figure 1. Series B depicts Newton's algorithm and series C depicts Circle approach algorithm

Example 2: Consider the function $f(x) = xe^x - 1$, $x \in R$. Then, minimizing point of the function is equal to -1 which is obtained in 10 iterations by Newton's algorithm and in 18 iterations by Circle approach algorithm (refer Table -2 and Figgure-2).

Remark : From the above examples, we observe that the minimizing sequential points of the given function obtained by Circle approach algorithm converges faster than Newton's algorithm to a very near point of the minimizing point initially and from that point to the actual minimizing point of the function it converges slowly or equally as Newton's algorithm.

Table-2

Iterations	Newton's algorithm	Circle approach algorithm
1	3	3
2	2.2	-0.99505
3	1.438095	-0.99677
4	0.728954	-0.99789
5	0.095395	-0.99862
6	-0.42737	-0.9991
7	-0.79149	-0.99941
8	-0.96403	-0.99961
9	-0.99875	-0.99975
10	-1	-0.99983
11		-0.99989
12		-0.99993
13		-0.99995
14		-0.99997
15		-0.99998
16		-0.99999
17		-0.99999
18		-1

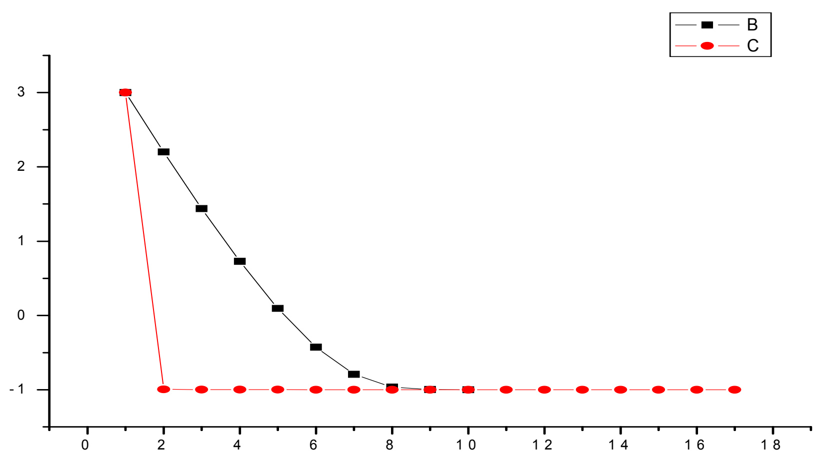

Figure 2. Series B depicts Newton's algorithm and series C depicts Circle approach algorithm

2.2 Circle-tangent approach algorithm

Now, we propose another new algorithm namely, Circle-tangent approach algorithm which is the combination of Circle approach algorithm and Newton's algorithm such that Circle approach algorithm is used initially for one or two iterations and then, Newton's algorithm is used for obtaining the minimizing point of the function using the approximate minimizing point by the Circle approach algorithm. The algorithm of Circle-tangent approach is given below.

Algorithm:

1. Given a non-linear function $f(x)$
2. Find a and b such that $f'(a)$ and $f'(b)$ are of opposite signs such that $a < b$.
3. Input $x_0, \in f'(x), f''(x), f'''(x)$
4. $n = 0$
5. If we choose $x_0 = a$, then go to step 6
   otherwise if we choose $x_0 = b$ then go to step 12.
6. Repeat
7. For $n = 1$, compute
$$x_{n+1} = x_n + \frac{f'(x_n)}{\sqrt{1 + f''^2(x_n) - f'(x_n) f'''(x_n)}}$$
8. For $n = 2,3,4\ldots$, compute $x_{n+1} = x_n - (f'(x_n)/f''(x_n))$
9. Until $|x_n - x_{n-1}| < \in$
10. Optimal solution $x^* \leftarrow x_n$
11. End
12. Repeat
13. For $n = 1$, compute
$$x_{n+1} = x_n - \frac{f'(x_n)}{\sqrt{1 + f''^2(x_n) - f'(x_n) f'''(x_n)}}$$
14. For $n = 2, 3, 4, \ldots$, compute $x_{n+1} = x_n - (f'(x_n)/f''(x_n))$
15. Until $|x_n - x_{n-1}| < \in$
16. Optimal solution $x^* \leftarrow x_n$
17. End

Convergence analysis for Circle tangent algorithm

This algorithm is a combination of the two algorithms namely, Circle approach algorithm and Newton's algorithm. For n = 1, we use Circle approach algorithm and for n = 2,3,4, … , we use Newton's algorithm. Since the order of convergence of Newton's method is 2, we have the order of convergence this method is 2.

A minimizing point for a given function by Newton's algorithm, Circle approach algorithm and Circle-tangent approach algorithm are obtained and the comparative study of the Newton's algorithm, Circle approach algorithm and Circle-tangent approach algorithm have been established by means of examples.

Example 3: Consider the function $f(x) = xe^{x} - 1$, $x \in R$. Then, minimizing point of the function is equal to -1 which is obtained by Newton's algorithm. in 10 iterations, Circle approach algorithm in 18 iterations and Circle-tangent approach algorithm in 4 iterations ((refer Table -3 and Figgure-3).

Table -3

Iterations	Newton's algorithm	Circle approach algorithm	Circle-tangent approach algorithm
1	3	3	3
2	2.2	-0.99505	-0.99505
3	1.438095	-0.99677	-0.99998
4	0.728954	-0.99789	-1
5	0.095395	-0.99862	
6	-0.42737	-0.9991	
7	-0.79149	-0.99941	
8	-0.96403	-0.99961	
9	-0.99875	-0.99975	
10	-1	-0.99983	
11		-0.99989	
12		-0.99993	
13		-0.99995	
14		-0.99997	
15		-0.99998	
16		-0.99999	
17		-0.99999	
18		-1	

3. Conclusion

In this paper, we have introduced new numerical algorithms namely, Circle approach algorithm and Circle-tangent approach algorithm for minimizing nonlinear unconstrained optimization problems. In real life problems, the variables can not be chosen arbitrarily rather they have to satisfy certain specified conditions called constraints. Such problems are known as constrained optimization problems. In near future, we have a plan to extend the proposed new algorithms namely, Circle approach algorithm and Circle-tangent approach algorithm to constrained optimization problems.

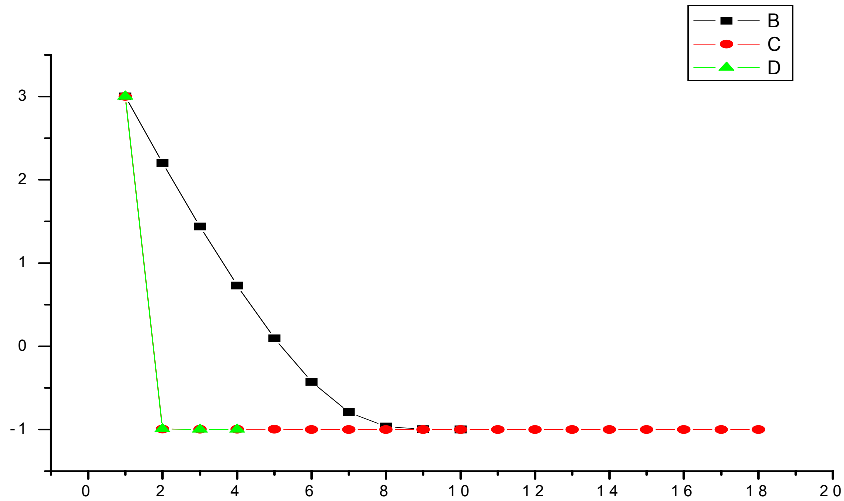

Figure 3. Series B depicts Newton's algorithm, series C depicts Circle approach algorithm and series D depicts Circle-tangent algorithm

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